SYMMETRIC SPACES
Frühlingssemester 2018

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Contents

CHAPTER 1

Introduction

The theory of symmetric spaces was initiated by E. Cartan in 1926. While he was studying Riemannian locally symmetric spaces, he discovered, via the paper by H. Weyl [**Wey26**], that the problem he was studying was equivalent to the one he had studied some twelve years earlier, namely the classification of real forms of complex semisimple Lie algebras.

The original definition of symmetric space belongs to the realm of Riemannian geometry, in that a Riemannian symmetric space was originally defined as a *Riemannian manifold whose curvature tensor is invariant under parallel translation.* While the Riemannian geometrical acception has not faded, Cartan discovered that symmetric spaces are as related to Riemannian geometry as they are to Lie groups.

There are at least three good reasons to study symmetric spaces:

- They connect seemingly different fields of mathematics, and hence each one of the fields can enhance the knowledge about the other. As Cartan put it: "The theory of groups and geometry, leaning on one another, allow one to take up and solve a great variety of problems", [**Car26**].
- Many well known examples are indeed symmetric spaces.
- They are beautiful!

Example 1.0.1. (1) The *Euclidean n-space* $\mathbb{E} := (\mathbb{R}^n, g_{Eucl})$ is a symmetric space. Its sectional curvature vanishes everywhere. Its isometry group is $\mathrm{O}(n) \ltimes \mathbb{R}^n$.

(2) The *unit sphere* S^n in $\mathbb{R}^{n+1}$ equipped with the Riemannian metric induced by $\mathbb{R}^{n+1}$ is a symmetric space whose sectional curvature is everywhere equal to one. Its isometry group is $\mathrm{O}(n, \mathbb{R})$.

(3) Let $q : \mathbb{R}^{n+1} \to \mathbb{R}$ be the quadratic form

$$q(x, y) := x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1} .$$

Then

$$\mathcal{H}^n_{\mathbb{R}} := \{x \in \mathbb{R}^{n+1} : q(x, x) = -1 \text{ and } x_{n+1} > 0\}$$

is the *(real)*[1] *hyperbolic n-space* and, equipped with the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$, it is a symmetric space whose sectional curvature

[1] We could define also the complex and quaternionic hyperbolic space. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Recall that the quaternions $\mathbb{H}$ is a four dimensional algebra over $\mathbb{R}$ with basis $\{1, i, j, k\}$, where 1 is central, $ij = k$, $jk = i$, $ki = j$, and $i^2 = j^2 = k^2 = -1$. Endow the space $\mathbb{K}^{n+1}$ with the

is identically equal to -1. Its isometry group is $\mathrm{O}(n,1)_+$, where

$$\mathrm{O}(n,1) := \{g \in \mathrm{GL}(n+1,\mathbb{R}) :\ q(gx,gy) = q(x,y) \text{ for every } x,y \in \mathbb{R}^{n+1}\}$$

and

$$\mathrm{O}(n,1)_+ := \{g \in \mathrm{O}(n,1) :\ g\mathcal{H}^n_{\mathbb{R}} = \mathcal{H}^n_{\mathbb{R}}\}\,.$$

In each of the above cases it is easy to see that the isometry group acts transitively on the symmetric space.

1.1. Overview

1.1.1. Riemannian Geometrical Characterization of Symmetric Spaces.

Convention. A manifold will always assumed to be connected, second countable, paracompact, Hausdorff and finite dimensional. The only exception are Lie groups, that are allowed to have several components.

If M is a Riemannian manifold and $p \in M$, a *geodesic symmetry* at p is a map defined in a neighborhood of p that fixes p and reverses any local geodesic through p.

Remark 1.1.1. A geodesic symmetry need not be an isometry and need not be defined on the whole of M.

Definition 1.1.2. The Riemannian manifold M is *Riemannian locally symmetric* if for every $p \in M$, there exists a geodesic symmetry s_p that additionally is an isometry on its domain.

A Riemannian locally symmetric space is *(globally) Riemannian symmetric* if in addition for every $p \in M$ the geodesic symmetry s_p is defined on the whole of M.

Example 1.1.3. (1) As an exercise define the geodesic symmetry in the case of S^n and of Euclidean n-space.
(2) Let $\mathcal{H}^n_{\mathbb{K}}$ be hyperbolic n-space. We can identify[2] the tangent space $T_x\mathcal{H}^n_{\mathbb{K}}$ at the point $x \in \mathcal{H}^n_{\mathbb{K}}$ with $x^\perp := \{y \in \mathbb{K}^{n+1} :\ q(x,y) = 0\}$ The Hermitian form q has signature $(n,1)$ and $\mathbb{K}^{n+1} = \mathbb{K}x \oplus \mathbb{K}x^\perp$, so that the restriction

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$\mathbb{K}$-Hermitian form q defined by

$$q(x,y) := \overline{x_1}y_1 + \dots + \overline{x_n}y_n - \overline{x_{n+1}}y_{n+1}\,,$$

(where conjugation is of course trivial in $\mathbb{R}$). If $\mathbb{PK}^n$ is the projective space $\mathbb{PK}^n = (\mathbb{K}^{n+1}\setminus\{0\})/\mathbb{K}^*$, the set

$$\mathcal{H}^n_{\mathbb{K}} := \{x \in \mathbb{PK}^n :\ q(x,x) < 0\}\,.$$

is called *real*, *complex* or *quaternionic hyperbolic n-space* $\mathcal{H}^n_{\mathbb{K}}$, according to whether $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Its dimension is, accordingly, n, $2n$ or $4n$.

[2]Consider the map $F(x) := q(x,x) + 1$. If $x \in F^{-1}(0)$, then $\ker d_xF = T_x(F^{-1}(0))$, and $(d_xF)(y) = \frac{d}{dt}|_{t=0}F(x+ty) = 2q(x,y)$.

of q to $x^\perp$ is positive definite: it follows that $\Re q(u,v)$ descends to an inner product on $T_x\mathcal{H}^n_{\mathbb{K}}$ that turns $\mathcal{H}^n_{\mathbb{K}}$ into a Riemannian manifold.

If for example $\mathbb{K}=\mathbb{R}$, then geodesics in this model are the intersection of the hyperboloid with planes through the origin. The geodesic symmetry is defined at x by

$$s_x(y) := -2xq(x,y) - y\,.$$

In fact, $s_x \in O(q,\mathbb{K})$, $(s_x)^2 = Id$, $s_x(x) = x$ and s_x preserves the Riemannian metric: namely, if $z \in \mathcal{H}^n_{\mathbb{K}}$, then $d_z s_x : T_z\mathcal{H}^n_{\mathbb{K}} \to T_{s_x(z)}\mathcal{H}^n_{\mathbb{K}}$ has the property that

$$q(d_z s_x(v), d_z s_x(v)) = q(s_x(v), s_x(v)) = q(v,v)\,,$$

where we have used that the differential of a linear map is the linear map itself. If follows also that, if v is a tangent vector at x, then

$$s_x(v) = 2vq(x,v) - v = -v\,.$$

If M is Riemannian symmetric, it is complete and the connected component of its isometry group is small enough to be finite dimensional, but large enough to act transitively. The stabilizer of a point is going to be a compact subgroup of $\mathrm{Iso}(M)^\circ$.

More (and less well known) examples:

Example 1.1.4.
(1) A compact semisimple Lie group can be turned into a Riemannian symmetric space.
(2) Any compact orientable Riemann surface of genus $g \geq 2$ is locally Riemannian symmetric but not Riemannian symmetric. They are all quotients $\mathcal{H}^2_{\mathbb{R}}/\Gamma$, where $\Gamma < \mathrm{Iso}(\mathcal{H}^2_{\mathbb{R}})^\circ$ is a discrete cocompact subgroup (isomorphic to the fundamental group of the surface).
(3) Quotients of 2-dimensional real hyperbolic space by $\mathrm{SL}(2,\mathbb{Z})$ or by any finite index subgroup are locally Riemannian symmetric with finite volume (but not necesarily compact).
(4) Borel showed that any Riemannian space, whose isometry group is semisimple, admits a quotient that is of finite volume and compact (using number theoretical arguments).

In fact, developing the theory leads to the first fact that any symmetric space is of the form $\mathbb{R}^m \times G/K$, where $\mathbb{R}^n$ is a Euclidean space, G is a semisimple group that has an involutive automorphism σ whose fixed point is essentially K (in fact, $(G^\sigma)^\circ \subset K \subset G^\sigma$).

1.1.2. Algebraic characterization of symmetric spaces. A symmetric space can be characterized from a purely algebraic point of view as follows. Take a connected Lie group and $\sigma : G \to G$ an involutive automorphism $\sigma^2 = Id$. A *symmetric space* for G is a homogeneous space G/H such that $H < G^\sigma$ is an open subgroup (hence union of connected components). If the group G^σ of σ-fixed points is compact, then G^σ can be equipped with a Riemannian metric by considering any G^σ-invariant inner product on the tangent space at eG^σ (which is possible since G^σ is compact) and smearing it around using the G-action. If $(G^\sigma)^\circ \leq K \leq G^\sigma$, then G/K is a Riemannian symmetric space.

REMARK 1.1.5. Differentiation of σ gives a decomposition of $\mathfrak{g}$ into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = Lie(H)$ is the eigenspace with eigenvalue $+1$ and $\mathfrak{m}$ is the eigenspace with eigenvalue -1. Then $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. These three conditions indeed are equivalent in turn to the existence of an involutive automorphism of G with $\mathfrak{h}$ as a $+1$ eigenspace and $\mathfrak{m}$ as a -1 eigenspace.

1.1.3. Equivalence Between the two Characterizations. If M is Riemannian symmetric, then $M \cong G/K$, where $G = \operatorname{Iso}(M)^\circ$ and $K = \operatorname{Stab}_G(p)$, where $p \in M$ is any point. Then K is compact and $\sigma : G \to G$, defined by $\sigma(g) = s_p g s_p$ is an involutive automorphism of G such that $(G^\sigma)^\circ \subset K \subset G^\sigma$ (and is hence open).

To see the converse, that is that $M = G/K$ is Riemannian symmetric, we need to define $s_p : M \to M$, where $p = hK \in M$. Then $s_p(lK) = h\sigma(h^{-1}l)K$, where σ is the involution of G fixing K. One can then see that $s_p(p) = p$, $s_p \in \operatorname{Iso}(M)$ and $d_p s_p : T_pM \to T_pM$ is just $d_p s_p = -Id$.

1.1.4. Decomposition and Classification. In 1926 Cartan classified all simply connected Riemannian symmetric spaces. Using the de Rham decomposition, one can see that any simply connected symmetric space can be written as a product of $M_0 \times M_+ \times M_-$, where

- M_0 has zero curvature and is hence isometric to $\mathbb{R}^n$;
- M_+ has non-negative sectional curvature;
- M_- has non-positive sectional curvature.

The simply connected symmetric spaces of non-negative curvature are those of *compact type*, while the M_- are of *non-compact type*. Both have semisimple isometry group. The compact and non-compact symmetric spaces are moreover dual one of the other (resembling the analogy between spherical geometry and hyperbolic geometry, that can be, in fact, explained by this duality).

An important invariant of a symmetric space is its *rank*. This can be explained from a Riemannian geometrical point of view as the maximal dimension of any totally geodesic subspace of M (that is the maximal dimension of a subspace of the tangent space to any point in which the curvature is zero). From the Lie theoretical point of view the rank is given in terms of the dimension of a Cartan subalgebra, that is a maximal abelian subalgebra that is diagonalizable.

If the rank is one, then the curvature is either negative or positive, and we have the hyperbolic spaces defined before (in negative curvature) and sphere (in positive curvature).

We will focus mostly on symmetric spaces of non-compact type (once we get there!). In this case $K < G$ is a maximal compact subgroup (and all maximal compact are conjugate) We will also see various decompositions, such as the Cartan and the Iwasawa decomposition. If time permits, we will study also the geometry at infinity of a symmetric space.

1.1.5. (Maximal) Prerequisites in Riemannian Geometry.

- Parallel transport, geodesic and the exponential map;
- Isometries of a Riemannian manifold as a metric space;
- de Rham decomposition;
- Levi-Civita connection;
- Curvatures (Riemann curvature tensor, sectional curvature);
- Jacobi fields.

1.1.6. Textbooks.

(1) A. Borel, [**Bor98**]
(2) M. Bridson and A. Haefliger, [**BH99**]
(3) M. do Carmo, [**dC92**]
(4) P. Eberlein, [**Ebe96**]
(5) S. Helgason, [**Hel01**]
(6) S. Kobayashi and K. Nomizu, [**KN96**]

CHAPTER 2

Generalities on Riemannian Globally Symmetric Spaces

2.1. Isometries and the Isometry Group

Definition 2.1.1. A map $f : M \to N$ between two Riemannian manifolds (M, g), (N, h) is an *isometry* if it is a diffeomorphism and $g = f^*h$, that is, if $d_p f : T_p M \to T_{f(p)} N$ is the differential, then

$$h_{f(p)}(d_p f(u), d_p f(v)) = g_p(u, v)\,,$$

for all $u, v \in T_p M$.

Note that an isometry maps geodesics into geodesics and hence preserves distances. The converse is also true, namely:

Theorem 2.1.2 ([**Hel01**, Theorem I.11.1]). *Any distance preserving self-map of the metric space (M, g) is an isometry.*

We also have the following very useful rigidity result:

Lemma 2.1.3 ([**Hel01**, Lemma I.11.2]). *Let $f_i : M \to N$, $i = 1, 2$, be two isometries of the connected Riemannian manifolds M and N. Suppose there exists a point $p \in M$ such that $f_1(p) = f_2(p)$ and $d_p f_1 = d_p f_2$. Then $f_1 = f_2$.*

Recall: The Riemannian exponential map at $p \in M$ is defined by $\mathrm{Exp}_p(X_p) := \gamma_{X_p}(1)$ in a sufficiently small neighborhood, where γ_{X_p} is the unique geodesic $\gamma : (-2, 2) \to M$ such that $\gamma(0) = p$ and $\dot\gamma_{X_p}(0) = X_p$. A neighborhood of $p \in M$ is called a *normal neighborhood* of p if U is the diffeomorphic image of a star shaped neighborhood on $0 \in T_p M$ under the Riemannian exponential map.

Proof of Lemma 2.1.3 . Let U be a normal neighborhood of $p \in M$ such that the $f_i|_U$ are diffeomorphisms. If $f := f_2^{-1} \circ f_1 : U \to U$, then $f(p) = p$ and $d_p f = Id$ and hence $f(y) = y$ for all $y \in U$. Since M is connected, any other point in M can be joined to p by overlapping normal neighborhoods. $\square$

The isometries of a Riemannian manifold (M, g) form a group $\mathrm{Iso}(M)$ under composition, that can be topologized with the compact-open topology (i.e. the topology generated by the subbasis $\{S(C, U) : C \subset M$ is compact and $U \subset M$ is open$\}$, where $S(C, U) := \{f \in \mathrm{Iso}(M) : f(C) \subset U\}$). Since M is a metric space, the compact-open topology is equivalent to the topology of the uniform convergence on compact sets [**Hel01**]. Since M is second countable and locally compact, $\mathrm{Iso}(M)$ is second countable as well (see [**Hel01**, Lemma IV.2.1]). Notice moreover that, by definition, $\mathrm{Iso}(M)$ acts effectively on M, that is there is no non-trivial subgroup of $\mathrm{Iso}(M)$ that leaves M invariant. In addition we have the following easy lemmas, that can be proven with straightforward verifications (as an exercise!)

LEMMA 2.1.4 ([**Hel01**, Lemma IV.2.3]). *If a sequence of isometries $(f_n)_n \in \mathrm{Iso}(M)$ converges pointwise on a set $S \subset M$, then it converges pointwise on the closure $\overline{S}$ of S.*

LEMMA 2.1.5 ([**Hel01**, Lemma IV.2.4]). *Let $(f_n)_n \in \mathrm{Iso}(M)$ be a sequence of isometries that converges pointwise on M to a map $f : M \to M$. Then the convergence is uniform on compact sets and $f \in \mathrm{Iso}(M)$.*

Using the above two lemmas, it is easy to prove the following:

THEOREM 2.1.6 ([**Hel01**, Theorem IV.2.2]). *Let $(f_n)_n \in \mathrm{Iso}(M)$ be a sequence that converges pointwise at one point $p_0 \in M$. Then there exists a subsequence $(f_{n_k}) \in \mathrm{Iso}(M)$ and $f \in \mathrm{Iso}(M)$, such that $f_{n_k} \to f$ uniformly on compact sets.*

The above Theorem 2.1.6 is one of the (at least) two ways to prove the following result:

THEOREM 2.1.7 ([**Hel01**, Theorem IV.2.5]). *Let M be a Riemannian manifold. Then $\mathrm{Iso}(M)$ is a locally compact group with the compact open topology and a topological group of transformation of M (i.e. it acts continuously on M). Moreover the stabilizer K of a point $p \in M$ is compact.*

PROOF. The first assertions are standard verifications and are left as an exercise. To see the compactness of the stabilizers, let $U \subset M$ a relatively compact neighborhood of p. By Theorem 2.1.6 the set $S(\{p\}, U)$ has relatively compact closure. Since $K \subset S(\{p\}, U)$ and K is closed, then K is compact. $\square$

The same assertion could have been obtained by using directly the Theorem of Ascoli-Arzelà (Theorem A.1.2).

We remark that $\mathrm{Iso}(M)$ is actually a finite dimensional Lie group, [**MS39**]. We will give later the (sketch of the) proof of this fact in the spacial case in which M is a Riemannian symmetric space.

2.2. Geodesic Symmetries

DEFINITION 2.2.1. Let M be a connected Riemannian manifold.

(1) M is *Riemannian locally symmetric* if for each $p \in M$ there exists a normal neighborhood U pf p and an isometry s_p on U that is:
   (a) involutive (i.e. $(s_p)^2 = Id$), and
   (b) p is an isolated fixed point (i.e. p is the only fixed point of s_p in U).
(2) M is *Riemannian (globally) symmetric* if for each $p \in M$, s_p can be extended to an isometry defined on M.

Here is the relation between Riemannian locally symmetric and Riemannian (globally) symmetric spaces.

THEOREM 2.2.2 ([**Hel01**, Theorem IV.5.6]). *A complete simply connected Riemannian locally symmetric spaces is Riemannian (globally) symmetric.*

We postpone the proof to later, after we will have proven that Riemannian symmetric spaces are analytic manifolds. This theorem implies in particular that the Riemannian universal cover of a Riemannian locally symmetric space is Riemannian globally symmetric. Conversely, every Riemannian locally symmetric space is a quotient of a Riemannian globally symmetric space by a discrete torsion-free group of isometries isomorphic to the fundamental group. However the converse of Theorem 2.2.2 does not hold, as for example S^1 is a globally symmetric space that is not simply connected.

In this course we will only be concerned with Riemannian globally symmetric spaces, so the terminology "Riemannian symmetric space" is from now on intended to mean Riemannian "globally symmetric space".

Since s_p is an isometry and hence it preserves geodesics (and whatever is defined in terms of geodesics), we have that the diagram

$$\begin{tikzcd} T_pM \arrow[r, "d_ps_p"] \arrow[d, "\mathrm{Exp}"'] & T_pM \arrow[d, "\mathrm{Exp}"] \\ M \arrow[r, "s_p"] & M\,. \end{tikzcd} \tag{2.2.1}$$

commutes. This will be used for example in the next lemma, which gives us an explicit expression for an involutive isometry.

LEMMA 2.2.3. *If M is a Riemannian manifold, $p \in M$ and s_p an involutive isometry of M, then $d_ps_p = -Id$ and $s_p(\operatorname{Exp} X_p) = \operatorname{Exp}(-X_p)$ for all $X_p \in T_p(M)$ for which* Exp *is defined.*

PROOF. Since s_p^2 is the identity, then $(d_ps_p)^2 = Id$, where $(d_ps_p)^2 : T_p(M) \to T_p(M)$. Hence d_ps_p has eigenvalues[1] $+1$ or -1. If $+1$ were to be an eigenvalue, then there would be $X_p \neq 0$ such that $d_ps_pX_p = X_p$. By (2.2.1), for small enough t, we would have that $s_p(\operatorname{Exp}(tX_p)) = \operatorname{Exp}(d_ps_p(tX_p)) = \operatorname{Exp}(tX_p)$. Thus the geodesic through p with initial vector X_p would be left fixed by s_p, contradicting the fact that p is an isolated fixed point of s_p.

Hence -1 must be the only eigenvalue and $d_ps_p = -Id$. Again by the commutativity of (2.2.1), we have that $s_p(\operatorname{Exp}(X_p)) = \operatorname{Exp}(d_ps_p(X_p)) = \operatorname{Exp}(-X_p)$, so that s_p reverses the geodesic through p with a given tangent vector. $\square$

The following corollary follows immediately from Lemma 2.2.3 and Lemma 2.1.3;

[1]Let V be a real vector space. Then any map $A \in \operatorname{End}(V)$ such that $A^2 = Id$ is diagonalizable. In fact, if $(\cdot,\cdot)$ is any inner product, then A is in the orthogonal group of the inner product $< u, v >:= (u, v) + (Au, Av)$ and hence is diagonalizable.

COROLLARY 2.2.4. *If M is a connected Riemannian manifold and $p \in M$, there is at most one involutive isometry s_p with p as isolated fixed point.*

The following lemma will allow us to draw the first very interesting consequence of the existence of geodesic symmetries.

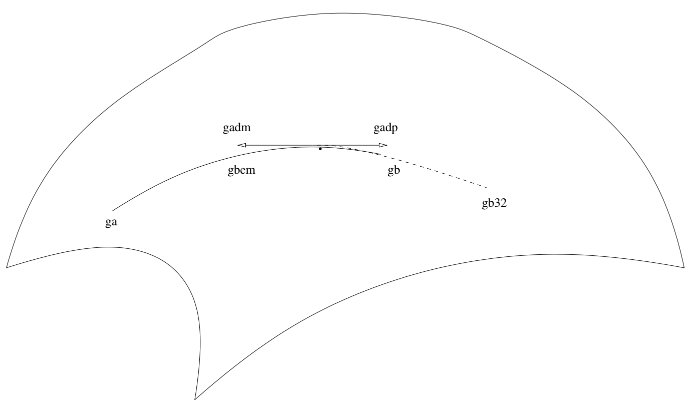
LEMMA 2.2.5. *Let M be a Riemannian symmetric space. Then the map $M \to \operatorname{Iso}(M)$ defined by $p \mapsto s_p$ is continuous.*

PROOF. We need to show that if $(p_n)_n \in M$ is a sequence such that $p_n \to p$, then $s_{p_n} \to s_p$ in the compact-open topology (that is, uniformly on compact sets). Because of Lemma 2.1.5, it is enough to show that $s_{p_n} \to s_p$ pointwise. Let $S \subset M$ be the set of points $x \in M$ for which $s_{p_n}(x) \to s_p(x)$. The set S is obviously not empty because the point $p \in S$ (since $d(p, s_{p_n}(p)) = 2d(p, p_n) \to 0$ as $n \to \infty$). By the smooth dependence on the initial conditions of solutions of differential equations (see Lemma A.2.1) the set S is open and by Lemma 2.1.4 S is closed. Hence $S = M$. □

PROPOSITION 2.2.6. *If M is a Riemannian symmetric space, then it is complete (as a metric space). Moreover the connected component* $\operatorname{Iso}(M)^\circ$ *of the isometry group* $\operatorname{Iso}(M)$ *acts transitively on M.*

PROOF. We will show that the geodesics are defined on $\mathbb{R}$. The first claim will then follow from the Theorem of Hopf–Rinow (Theorem A.2.2).

Let γ be a geodesic in M and assume that γ is defined on some interval $[a, b)$ with $a < b$. We will show that it can be extended to b. Take $\epsilon = \frac{b-a}{4}$ and consider the geodesic symmetry s_{p_0} at $p_0 := \gamma(b - \epsilon)$.



It takes the geodesic $\gamma(t)$ to another geodesic through p_0 whose tangent vector at p_0 is $-\dot\gamma(b-\epsilon)$ and whose length is the same as that of $\gamma(t)$. Since the tangent vectors at the point p_0 are the same (up to a scalar), then the new geodesic coincides with γ for all $\frac{a+b}{2} < t < b$ and thus extends it to the interval $[a, \frac{3}{2}b] \supsetneq [a, b]$.

By the Theorem of Hopf–Rinow, given any two points $p, q \in M$ with $t := d(p, q)$, there exists a geodesic $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\gamma(t) = q$. Then $q = s_{\gamma(t/2)}(p)$, that is $\operatorname{Iso}(M)$ acts transitively.

By Lemma 2.2.5, the isometries s_p are in the same connected component for all $p \in M$, but this connected component is not necessarily the one that contains the identity. On the other hand, the set $\{s_p \circ s_{p'}\}$ is contained in $\mathrm{Iso}(M)^\circ$, since $(s_p)^2 = Id$ and the map $p \mapsto s_p$ is continuous, which allows us to deform s_p^2 to $\{s_p \circ s_{p'}\}$ by any path from p to p'. Since $q = s_{\gamma(t/2)} \circ s_p(p)$, we have shown that $\mathrm{Iso}(M)^\circ$ acts transitively on M. $\square$

A classical theorem of Myers and Steenrod [**MS39**] assert that the isometry group of a Riemannian manifold is a Lie group. The idea is to consider orbits of points and parametrize in this way $\mathrm{Iso}(M)$. We sketch the proof in the special case of our interest.

THEOREM 2.2.7. *Let M be a Riemannian globally symmetric space. Then $G := \mathrm{Iso}(M)$ has a Lie group structure compatible with the compact open topology and it acts smoothly on M. Moreover M is diffeomorphic to G/K, where $K = \mathrm{Stab}_G(p_0)$ (for $p_0 \in M$ a base point) is compact and contains no non-trivial normal subgroups of G.*

SKETCH OF THE PROOF. The map $K \to \mathrm{O}(T_{p_0}M, g)$, defined by $k \mapsto d_{p_0}k$, is a homeomorphism onto its image. Hence K can be identified with a closed subgroup of $\mathrm{O}(T_{p_0}M, g)$, from which it inherits a unique differentiable structure compatible with the topology, which makes it a Lie group.

Let $\pi : G \to M$ be the natural projection, $\pi(g) := g \cdot p_0$. We will construct a continuous local section of π, that is a map $\phi : B_r(p_0) \to G$, where $B_r(p_0)$ is a normal ball in M, such that $\pi \circ \phi = Id$ (see Definition A.1.3). From this it will follow that $\pi|_B$ is a homeomorphism, where $B := \phi(B_r(p_0))$, and hence $\pi^{-1}(B_r(p_0)) = BK = \{bk : b \in B, k \in K\}$ is an open set in G homeomorphic to $B \times K$. The differentiable structure will hence be given to G by using translates of open set BU, where $U \subset K$ is open and one can check that all the needed properties hold.

In order to construct the section ϕ, let $\gamma(t)$ be a geodesic in $B_r(p_0)$ such that $\gamma(0) = p_0$. As seen already in the proof of Proposition 2.2.6, for every t, the isometry $s_{\gamma(t/2)} \circ s_{p_0}$ maps p_0 into $\gamma(t)$. Define $\phi(\gamma(t)) := s_{\gamma(t/2)} \circ s_{p_0}$. The map ϕ has the desired properties, since it is obviously injective for small enough t and continuous (Lemma 2.2.5).

If K were to contain a subgroup that is normal in G, then this subgroup would act trivially on $M = G/K$, which is impossible. $\square$

REMARK 2.2.8. The smooth structure on a Lie group is in fact analytic. This follows for example from the fact that the exponential map on an arbitrary Riemannian manifold is analytic and hence so are the coordinate systems it defines. In particular, the same proof that shows that there is only one smooth structure on a Lie group compatible with the topology with respect to which it is a Lie group, shows that there is only one analytic structure. A consequence of the existence of the analytic structure is that all above statements are analytic and not only smooth. This is useful to prove Theorem 2.2.2, whose proof we sketch here.

Recall that the difference between a locally symmetric space and a globally symmetric space is whether or not the geodesic symmetries are globally defined. The purpose of the following lemmas will hence to provide such an extension in the case of a simply connected locally symmetric space. The first one will use that if two analytic maps defined on an open set V coincide on an open subset $U \subset V$, $f_1|_U = f_2|_U$, then they coincide on the larger open set, $f_1 \equiv f_2$ on V. The next two lemmas will allow to extend an isometry along a path and to show that the extension does not depends on the choice of a curve in a given homotopy class. One can then wrap all of this up to prove Theorem 2.2.2.

Lemma 2.2.9. *Let M and N be complete Riemannian real analytic manifolds and let $B_\rho(p)$ be a normal ball around $p \in M$. Let $f : B_r(p) \to B_r(f(p))$ be an isometry, with $r < \rho$ and $B_r(f(p))$ a normal neighborhood of $f(p)$. Then f is analytic and it extends to an isometry $f : B_\rho(p) \to B_\rho(f(p))$.*

Proof. By the commutativity of the diagram

$$\begin{tikzcd} T_pM \arrow[r, "d_pf"] \arrow[d, "\mathrm{Exp}_p"'] & T_qN \arrow[d, "\mathrm{Exp}_q"] \\ B_r(p) \arrow[r, "f"] & N \end{tikzcd} \tag{2.2.2}$$

it follows that $f = \mathrm{Exp}_q \circ d_pf \circ \mathrm{Exp}_p^{-1}$ is analytic on $B_r(p)$ and hence can be extended to $f' : B_\rho(p) \to N$. We still need to verify that f' is an isometry. To this purpose, let X, Y be analytic vector fields on $B_\rho(p)$. By assumption, if g, h are the Riemannian metrics on M, N respectively, we have that on $f^*h = g$ on $B_r(p)$, that is

$$h_q(d_pf'X_p, d_pf'Y_p) = g(X_p, X_p)\,.$$

This equality holds by assumption on $B_r(p)$. By analyticity, the equality holds on $B_\rho(p)$ and hence f' is an isometry on $B_\rho(p)$. □

Lemma 2.2.10 ([**Hel01**, Proposition I.11.3]). *Let M and N be complete real analytic Riemannian manifolds. Let $p \in M$ and $f : U \to N$ be an isometry, where $U \subset M$ is a normal neighborhood of p. Let η be any curve in M starting at p. Then f can be continued along η, i.e. for each $t \in [0,1]$ there exists a neighborhood U_t of $\eta(t)$ and an isometry $f_t : U_t \to N$, such that:*

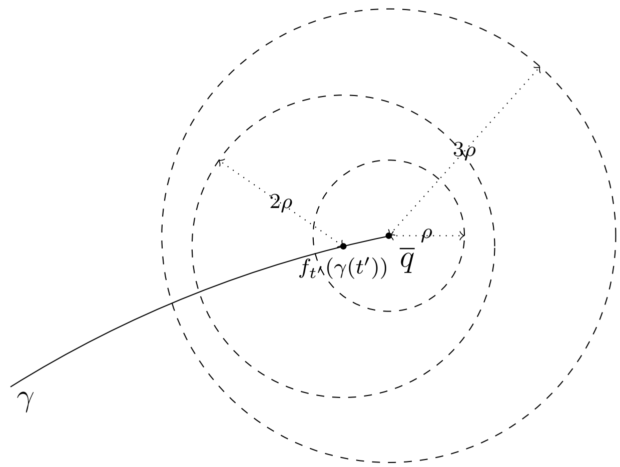
(1) *$U_0 = U$, $f_0 = f$, and*
(2) *there exists $\epsilon > 0$, such that for $|t - s| < \epsilon$, $U_t \cap U_s \neq \emptyset$ and $f_t = f_s$ on $U_t \cap U_s$.*

Proof. Let $I := \{t \in [0,1] :\ f$ can be extended near $\eta(t)\}$. Then I is an open interval and it is not empty since $0 \in I$. We will show that $\bar{t} := \sup I \in I$, so that $I = [0,1]$.

Let $\overline{q} := \lim_{t\to\bar{t}} f_t(\eta(t))$. This exists because N is complete. Choose $\rho > 0$ so that $B_{3\rho}(\eta(\bar{t}))$ and $B_{3\rho}(\overline{q})$ are normal balls around $\eta(\bar{t})$ and $\overline{q}$ respectively. Let $t' < \bar{t}$ such that

$$\eta(t') \in B_\rho(\eta(\bar{t})) \qquad \text{and} \qquad f_t(\eta(t')) \in B_\rho(\overline{q})$$

for all $t' \leq t < \bar{t}$.



Then $B_{2\rho}(\eta(t'))$ and $B_{2\rho}(f_t(\eta(t')))$ are normal balls around $\eta(t')$ and $f_t(\eta(t'))$ respectively and, by the first part of the argument, $f_{t'}$ can be extended. Since $\eta(\bar{t}) \in B_{2\rho}(\eta(t'))$, then $\bar{t} \in I$. □

REMARK 2.2.11. Since $f_t(\eta(t))$ and $d_{\eta(t)}f_t$ vary continuously with t and N is Hausdorff, the continuation is unique. In fact, let f'_t be another continuation of f along $\eta(t)$ for $t \in [0,1]$. Then the set of $t \in [0,1]$ for which $f_t(\eta(t)) = f'_t(\eta(t))$ and $d_{\eta(t)}f_t = d_{\eta(t)}f'_t$ is an open and closed subset of $[0,1]$ that contains 0 and is hence the whole interval $[0,1]$.

LEMMA 2.2.12. *Let M and N be complete Riemannian locally symmetric spaces. Let $p \in M$ and $f : U \to N$ be a isometry, where $U \subset M$ be a normal neighborhood of p. Let η be any curve in M starting at p and δ another curve, homotopic to η with fixed endpoints. If f^η and f^δ are the continuation of f along η and δ, then f^η and f^δ agree in a neighborhood of $\eta(1) = \delta(1)$.*

PROOF. Let $H : [0,1] \times [0,1] \to M$ be a homotopy between η and δ, with $H(t,0) = \eta(t)$, $H(t,1) = \delta(t)$, $H(0,s) = p$, $H(1,s) = \eta(1) = \delta(1)$.

Call f^s the continuation of f along the curve $H_s : t \mapsto H(t,s)$ (see Lemma 2.2.10). Define $I := \{s \in [0,1] :$ for all $s' \leq s$, $f^{s'}(1) = f^0(1) = f^1(1)$ near $\eta(1) = \delta(1)\}$. Then I is open and not empty (since it contains 0). We will show that $\varsigma := \sup I \in I$, thus showing that $I \subset [0,1]$ is open, closed and not empty (and hence $I = [0,1]$).

Since the maps $t \mapsto H_\varsigma(t)$ and $t \mapsto f^\varsigma(H_\varsigma(t))$ are continuous, there exists $\rho > 0$ such that $B_{2\rho}(H_\varsigma(t))$ and $B_{2\rho}(f^\varsigma(H_\varsigma(t)))$ are normal balls for all $0 \leq t \leq 1$. By definition of ς, then there exists $\epsilon > 0$ such that $d(H_\varsigma(t), H_s(t)) \leq \rho$ for all t and

$|\varsigma - s| < \epsilon$. Then f^ς is a continuation of f along H_s and therefore, by uniqueness, $f^\varsigma = f^s$ near $\eta(1) = \delta(1)$. This concludes the proof that $\varsigma \in I$. □

Proof of Theorem 2.2.2. Let $p \in M$ and let $B_r(p)$ be a normal ball such that the geodesic symmetry s_p is an isometry of $B_r(p)$ into itself. We are going to define a map $\Phi : M \to M$ that is an isometry, coincides with s_p on $B_r(p)$ and is involutive with isolated fixed points, thus proving that M is Riemannian globally symmetric. Let $q \in M$ be any point, and let $\gamma : [0,1] \to M$ be a continuous path such that $\gamma(0) = p$ and $\gamma(1) = q$. Let us continue s_p along γ and let us define $\Phi(q) := (s_p)_1(1)$ (where, in the notation of Lemma 2.2.10, $(s_p)_t$ denotes the extension of s_p in a neighborhood of $\gamma(t)$). By Lemma 2.2.12, since M is simply connected, $\Phi(q)$ does not depend on the choice of the path γ and coincides with $(s_p)_1$ in a neighborhood of $\gamma(1)$. Hence $\Phi : M \to M$ is an differentiable map whose differential preserves the Riemannian metric. Since it reverses geodesics at p, then $\Phi^2 = Id$. It follows that Φ is an isometry. □

2.3. Transvections and Parallel Transport

We saw in the proof of Proposition 2.2.6 that the set of geodesic symmetries is transitive on a Riemannian globally symmetric space. In particular, we saw that if $p, q \in M$ and $\gamma : \mathbb{R} \to M$ is geodesic such that $\gamma(0) = p$ and $\gamma(t) = q$, then $q = s_{\gamma(t/2)} \circ s_{\gamma(0)}(p)$.

Proposition 2.3.1. *Let M be a Riemannian globally symmetric space, $\gamma : \mathbb{R} \to M$ a geodesic and $\mathcal{T}_t := s_{\gamma(t/2)} \circ s_{\gamma(0)}$. Then for every $c \in \mathbb{R}$,*

$$\mathcal{T}_t(\gamma(c)) = \gamma(t + c)\,.$$

Moreover $d_{\gamma(0)}\mathcal{T}_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ is the parallel translation along the geodesic γ, that is, if $X_{\gamma(0)} \in T_{\gamma(0)}M$, then $X_{\gamma(t)} := d_{\gamma(0)}\mathcal{T}_t X_{\gamma(0)}$ is the associated parallel vector field along γ.

The first assertion of the above proposition explains the reason for the following terminology.

Definition 2.3.2. The isometries $\mathcal{T}_t := s_{\gamma(t/2)} \circ s_{\gamma(0)}$ are called *transvections*.

Proof of Proposition 2.3.1. Since geodesic symmetries map geodesic onto themselves changing the orientation, the map $\mathcal{T}_t$ must map the geodesic γ onto itself and preserve its orientation. If we assume that γ is a unit speed parametrization, it follows that the restriction to the geodesic $\gamma(t)$ has the form $\mathcal{T}_t(\gamma(c)) = \gamma(c + \text{constant})$. Since $\mathcal{T}_t(\gamma(0)) = \gamma(t)$, then $\mathcal{T}_t(\gamma(c)) = \gamma(t + c)$.

We now consider the action of $d_{\gamma(0)}\mathcal{T}_t$ on $T_{\gamma(0)}M$. Let $v := X_{\gamma(0)} \in T_{\gamma(0)}M$ and let X^v be the unique parallel vector field such that $(X^v)_{\gamma(0)} = X_{\gamma(0)}$. We want to show that $(X^v)_{\gamma(t)} = (d_{\gamma(0)}\mathcal{T}_t)X_{\gamma(0)}$for all $t \in \mathbb{R}$.

Since for every t $s_{\gamma(t')}$ is an isometry and X^v is parallel along γ, then by Lemma A.2.9 $((s_{\gamma(t')})_*)X^v$ is a vector field parallel along $s_{\gamma(t')} \circ \gamma = \gamma$. At the point $\gamma(t')$ the value of this new parallel vector field is

$$(s_{\gamma(t')})_*(X^v)_{\gamma(t')} = -(X^v)_{\gamma(t')} \,.$$

But $-X^v$ is also parallel along γ with value $-(X^v)_{\gamma(t')}$ at $\gamma(t')$. By uniqueness of parallel vector fields with prescribed initial conditions we have

$$(s_{\gamma(t')})_* X^v = -X^v \,. \tag{2.3.1}$$

Observe now that

$$s_{\gamma(t')}(\gamma(t)) = \gamma(2t'-t)\,,$$

so that (2.3.1) can be written as

$$(d_{\gamma(t)} s_{\gamma(t')})(X^v)_{\gamma(t)} = (-X^v)_{\gamma(2t'-t)} \,. \tag{2.3.2}$$

At $t'=0$ (2.3.2) becomes

$$(d_{\gamma(t)} s_{\gamma(0)})(X^v)_{\gamma(t)} = (-X^v)_{\gamma(-t)}\,, \tag{2.3.3}$$

while at $t'=c/2$ with $-t$ replacing t, (2.3.2) becomes

$$(d_{\gamma(-t)} s_{\gamma(c/2)})(X^v)_{\gamma(-t)} = (-X^v)_{\gamma(c+t)}\,, \tag{2.3.4}$$

Now

$$\begin{aligned} d_{\gamma(t)}\mathcal{T}_c(X^v)_{\gamma(t)} &= d_{\gamma(t)}(s_{\gamma(c/2)} \circ s_{\gamma(0)})(X^v)_{\gamma(t)} \\ &= (d_{\gamma(-t)} s_{\gamma(c/2)})(d_{\gamma(t)} s_{\gamma(0)})(X^v)_{\gamma(t)} \\ &\overset{(2.3.3)}{=} (d_{\gamma(-t)} s_{\gamma(c/2)})(-(X^v)_{\gamma(-t)}) \\ &\overset{(2.3.4)}{=} (X^v)_{\gamma(c+t)} \,. \end{aligned}$$

By setting $t=0$ and $c=t$, we obtain the desired equality. $\square$

Remark 2.3.3. The same argument as above shows that

$$s_{\gamma(t_1)} \circ s_{\gamma(t_2)}(\gamma(t)) = \gamma(t + 2(t_1 - t_2))$$

and $d_{\gamma(t)}(s_{\gamma(t_1)} \circ s_{\gamma(t_2)})$ is the parallel transport along $\gamma(t)$.

2.4. Algebraic Point of View

We have seen that if M is Riemannian (globally) symmetric, then M is diffeomorphic to G/K, where $G = \mathrm{Iso}(M)^\circ$ and K is the stabilizer of a point in M. In this section we will deal with the natural question regarding the converse statement: which homogeneous spaces are Riemannian symmetric spaces?

Let G be a Lie group with Lie algebra $\mathfrak{g}$. The Lie group exponential map $\exp : \mathfrak{g} \to G$ is defined as $\exp_{\mathfrak{g}}(X) := \varphi_x(1)$, where $\varphi_X : \mathbb{R} \to G$ is the unique one-parameter subgroup (that is a homomorphism from $\mathbb{R}$ into G such that $\dot\varphi_X(0) = X$.

If $h : G \to G$ is any homomorphism, then by naturality of the Lie group exponential map $\exp : \mathfrak{g} \to G$, the following diagram commutes

(2.4.1)
$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d_e h"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "h"'] & G \end{tikzcd},$$
that is, for all $g \in G$ and $X \in \mathfrak{g}$,
$$\exp d_e h X = h(\exp X)\,.$$

A particularly important homomorphism is the conjugation $c_g : G \to G$, defined by $c_g(h) := ghg^{-1}$. This is a Lie group isomorphism whose differential at the identity $e \in G$, $\mathrm{Ad_G}(g) := d_e c_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism. In particular

$$\mathrm{Ad_G}(g)([X,Y]) = [\mathrm{Ad_G}(g)(X), \mathrm{Ad_G}(g)(Y)]\,,$$

for all $X, Y \in \mathfrak{g}$ and $g \in G$. The map $\mathrm{Ad_G} : G \to \mathrm{GL}(\mathfrak{g})$ is an analytic group homomorphism called the *adjoint representation of* G. Its derivative at the identity $d_e\mathrm{Ad_G} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the *adjoint representation of* $\mathfrak{g}$ and is denoted by $\mathrm{ad}_{\mathfrak{g}} := d_e\mathrm{Ad_G}$.So for example (2.4.1) applied to $h = c_g$ yields

$$\exp \mathrm{Ad_G}(g)X = g \exp X g^{-1}\,.$$

If $\sigma \in \mathrm{Aut}(G)$, we set $G^\sigma := \{g \in G : \sigma(g) = g\}$.

DEFINITION 2.4.1. Let G be a connected Lie group. An automorphism $\sigma \in \mathrm{Aut}(G)$ is *involutive* if $\sigma^2 = Id$.

DEFINITION 2.4.2. Let G be a connected Lie group and $K \leq G$ a closed subgroup. The pair (G, K) is called *a Riemannian symmetric pair* if:

(1) $\mathrm{Ad_G}(K)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$, and
(2) there is an analytic involutive automorphism σ of G such that

$$(G^\sigma)^\circ \subset K \subset G^\sigma\,.$$

The prominent example of Riemannian symmetric pair is given by the next result.

PROPOSITION 2.4.3. *Let M be a Riemannian symmetric space and $G := \mathrm{Iso}(M)^\circ$. Fix a base point $p \in M$ and let K be the isotropy subgroup of G at p. Then the map $\sigma : G \to G$ defined by $g \mapsto s_p g s_p$ is an involutive Lie group automorphism of G such that $(G^\sigma)^\circ \subset K \subset G^\sigma$.*

It makes sense to give then the following:

DEFINITION 2.4.4. Let M be a Riemannian (globally) symmetric space, $G = \mathrm{Iso}(M)^\circ$ and $K \leq G$ the isotropy subgroup of a point $p \in M$. Then (G, K) is called *Riemannian symmetric pair associated to* (M, p).

Notice that one cannot say anything more precise of the relation between K and G^σ, as the following examples show:

EXAMPLE 2.4.5. (1) Let $M = S^2$, $p = e_3$ and $G = \mathrm{Iso}(M) = \mathrm{SO}(3,\mathbb{R})$. We can write s_p and $g \in \mathrm{SO}(3,\mathbb{R})$ in block form as

$$s_p = \begin{pmatrix} Id_2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} A & b \\ c & d \end{pmatrix},$$

so that

$$\sigma(g) = \begin{pmatrix} Id_2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & b \\ c & d \end{pmatrix} \begin{pmatrix} Id_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & -b \\ -c & d \end{pmatrix}$$

and hence

$$G^\sigma = \{g \in \mathrm{SO}(3,\mathbb{R}) : \mathrm{g} = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \text{ with } \mathrm{A} \in \mathrm{O}(2,\mathbb{R}), \mathrm{d} = \pm 1, (\det \mathrm{A})\mathrm{d} = 1\}$$

has two connected components. In this case we have also that K is connected since S^2 is simply connected[2], so that $(G^\sigma)^\circ = K \subsetneq G^\sigma$ and

$$K = \{g \in \mathrm{SO}(3,\mathbb{R}) : \mathrm{g} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \text{ with } \mathrm{A} \in \mathrm{SO}(2,\mathbb{R})\}.$$

(2) If $M = \mathbb{P}(\mathbb{R}^3) = S^2/\{\pm Id\}$, then $p = [e_3]$, $G = \mathrm{Iso}(M)^\circ = \mathrm{Iso}(M) = \mathrm{O}(3,\mathbb{R})/\pm Id$ and $(G^\sigma)^\circ = K = G^\sigma$.

PROOF OF PROPOSITION 2.4.3. First we verify that $g \mapsto s_p g s_p$ is involutive. In fact, since s_p^2 is the identity,

$$\sigma^2(g) = \sigma(\sigma(g)) = \sigma(s_p g s_p) = s_p(s_p g s_p)s_p = s_p^2 g s_p^2 = g\,.$$

We verify now that $K \subset G^\sigma$, that is that for every $k \in K$, $\sigma(k) = s_p k s_p = k$. To see this observe first of all that $\sigma(k)(p) = (s_p k s_p)(p) = s_p(k(s_p(p))) = s_p(k(p)) = s_p(p) = p$. Moreover, as $d_p\sigma(k) : T_pM \to T_pM$ and $d_p s_p = -Id$, we have that $d_p\sigma(k) = d_p(s_p k s_p) = d_p s_p \circ d_p k \circ d_p s_p = d_p k$, By Lemma 2.1.3, $\sigma(k) = k$, that is $K \subset G^\sigma$.

Conversely, to show that $(G^\sigma)^\circ \subset K$, it is enough to see that K contains a neighborhood of the identity in G^σ. Let $V \subset M$ be an open neighborhood of p: by continuity, there exists an open neighborhood $U \subset G^\sigma$ of e such that $g(p) \in V$ for all $g \in U$. But if $g \in U \subset G^\sigma$, then $s_p g s_p = g$, so that $g(p) \in V$ is a fixed point of s_p, $s_p g(p) = g s_p(p) = g(p)$. Since s_p has only isolated fixed points, we could have chosen V in such a way that p is the only fixed point of s_p in M, which would imply that $g(p) = p$. Thus $U \subset K$. □

The following theorem answers in particular the question at the beginning of § 2.4.

[2]If G is a connected topological group, $H \leq G$ a closed subgroup such that G/H is simply connected, then H is connected. In fact, let H° be the connected component of the identity of H. Then $G/H^\circ \to G/H$ is a covering map. Moreover, since G is connected, then G/H is connected. Since G/H is simply connected, the covering map must be the identity.

THEOREM 2.4.6. *If (G, K) is a Riemannian symmetric pair and σ is an involutive automorphism of G such that $(G^\sigma)^\circ \subset K \subset G^\sigma$, then G/K is a globally symmetric space with respect to any G-invariant Riemannian metric. If $\pi : G \to G/K$ denotes the natural projection and s_p the geodesic symmetry at $p = \pi(K) = eK \in G/K$, then*

$$s_p \circ \pi = \pi \circ \sigma\,.$$

In addition the geodesic symmetry s_p is independent of the choice of the G-invariant metric for any $p \in G/K$.

We start the proof of Theorem 2.4.6 with few lemmas that it is good to emphasize.

LEMMA 2.4.7. *Let (G, K) be a Riemannian symmetric pair with an involutive automorphism σ, and let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebra of G and K respectively. Then*

(1) *$\mathfrak{k} = \{X \in \mathfrak{g} : d_e\sigma X = X\}$, and*
(2) *if $\mathfrak{p} := \{X \in \mathfrak{g} : d_e\sigma X = -X\}$, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

PROOF. We verify first that $\mathfrak{k}$ is the set of fixed points of $d_e\sigma$. By definition of symmetric pair $\dim(G^\sigma)^\circ = \dim K = \dim(G^\sigma)^\circ$ so that, if $\mathfrak{k}$ is the Lie algebra of K,

$$\begin{aligned}
\mathfrak{k} =& \operatorname{Lie}(G^\sigma)\\
=&\{X : \exp tX \in G^\sigma \text{ for all } t \in \mathcal{R}\}\\
=&\{X : \sigma(\exp tX) = \exp tX \text{ for all } t \in \mathcal{R}\}\\
\overset{(2.4.1)}{=}&\{X : d_e\sigma X = X\}\,.
\end{aligned}$$

To see the second assertion we write $X = \frac{1}{2}(X + d_e\sigma X) + \frac{1}{2}(X - d_e\sigma X)$. Since $(d_e\sigma)^2 = Id$, then $\frac{1}{2}(X + d_e\sigma X) \in \mathfrak{k}$ and $\frac{1}{2}(X - d_e\sigma X) \in \mathfrak{p}$. □

LEMMA 2.4.8. *Let (G, K) be a Riemannian symmetric pair with an involutive automorphism σ, and let $\mathfrak{p} := \{X \in \mathfrak{g} : d_e\sigma X = -X\}$. Then $\mathfrak{p}$ is $\mathrm{Ad_G}(K)$-invariant.*

PROOF. We need to show that for all $X \in \mathfrak{p}$ and all $k \in K$

$$d_e\sigma(\mathrm{Ad_G}(k)(X)) = -\mathrm{Ad_G}(k)(X)\,.$$

To this purpose, observe first of all that

(1) from the naturality of the exponential map, $\exp \circ d_e\sigma(tX) = \sigma(\exp(tX))$ for all $X \in \mathfrak{g}$. Inn particular if $X \in \mathfrak{p}$, then

$$\exp(-tX) = \sigma(\exp(tX))\,. \tag{2.4.2}$$

(2) Moreover, again for the naturality of the exponential map and the definition of $\mathrm{Ad_G}$, we have that for all $X \in \mathfrak{g}$

$$\exp(\mathrm{Ad_G}(k)(tX) = c_k(\exp(tX)) = k(\exp(tX))k^{-1}\,.$$

But if $X \in \mathfrak{p}$, using (2.4.2) and the fact that $K \subset G^\sigma$, we have that

$$\begin{aligned} \exp(\mathrm{Ad}_{\mathrm{G}}(k)(tX) =& c_k(\exp(tX)) = k(\exp(tX))k^{-1} \\ =& k\sigma(\exp(-tX))k^{-1} = \sigma(k\exp(-tX)k^{-1})\,. \end{aligned} \tag{2.4.3}$$

(3) Finally, from the commutativity of the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_{\mathrm{G}}(k)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[r, "d_e\sigma"] \arrow[d, dashed] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "c_k"'] & G \arrow[r, "\sigma"'] & G\,, \end{tikzcd} \tag{2.4.4}$$

we have that

$$\exp d_e\sigma \mathrm{Ad}_{\mathrm{G}}(k)(tX) = \sigma(c_k \exp(tX)) = \sigma(k\exp(tX)k^{-1})\,.$$

By equating the right hand side of this formula and of (2.4.3), replacing tX with $-tX$, we obtain

$$\exp(\mathrm{Ad}_{\mathrm{G}}(k)(-tX) = \exp d_e\sigma\mathrm{Ad}_{\mathrm{G}}(k)(tX)$$

Since exp is locally injective, we deduce that $\mathrm{Ad}_{\mathrm{G}}(k)(-tX) = d_e\sigma\mathrm{Ad}_{\mathrm{G}}(k)(tX)$, for all $X \in \mathfrak{p}$, $k \in K$.

□

Proof of Theorem 2.4.6. If $\pi : G \to G/K$ is the projection and we set $\pi(e) =: p \in G/K$, then the differential $d_e\pi : \mathfrak{g} \to T_p(G/K)$ is surjective and has kernel $\ker d_e\pi = \mathfrak{k}$, so that there is an isomorphism of $\mathbb{R}$-vector spaces $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{k} \cong T_p(G/K)$. We will see now that $d_e\pi$ intertwines the action of K via Ad_{G} on $\mathfrak{p}$ with the action on $T_p(G/K)$ obtained by differentiating the left translation by k. In fact, by the commutativity of the left diagram in (2.4.4), and applying π on both sides, we have that

$$\pi(\exp(\mathrm{Ad}_{\mathrm{G}}(k)(tX))) = \pi(k\exp(tX)k^{-1}) = k\pi(\exp(tX)k^{-1}) = k\pi(\exp(tX))\,.$$

By taking the derivative at the origin $t = 0$, we obtain the desired intertwining of the actions

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "\mathrm{Ad}_{\mathrm{G}}(k)"] \arrow[d, "d_e\pi"'] & \mathfrak{p} \arrow[d, "d_e\pi"] \\ T_o(G/K) \arrow[r, "d_ek"'] & T_o(G/K) \end{tikzcd}$$

that is $d_e\pi \circ \mathrm{Ad}_{\mathrm{G}}(k)(X) = d_pk \circ d_e\pi(X)$ for all $X \in \mathfrak{p}$ and all $k \in K$.

Since $\mathrm{Ad}_{\mathrm{G}}(K)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$, there exists a positive definite inner product B on $\mathfrak{p}$ and, actually, any positive definite inner product can be made

$\mathrm{Ad}_G(K)$ invariant. In fact, if $B' : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ be a positive definite inner product on $\mathfrak{p}$ and μ is the Haar measure on $\mathrm{Ad}_{\mathrm{G}}(K)$, then

$$B(X,Y) := \int_{\mathrm{Ad}_{\mathrm{G}}(K)} B'(g_* X, g_* Y)\, d\mu(k)$$

is obviously $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant and can be proven to be non-zero.

We set now $Q_p := B \circ (d_e\pi^{-1} \times d_e\pi^{-1}) : T_p(G/K) \times T_p(G/K) \to \mathbb{R}$, which is now a K-invariant inner product on $T_p(G/K)$ and we extend it to $T_r G/K$ by invariance. Namely, if $X_r, Y_r \in T_r(G/K)$, then

$$Q_r(X_r, X_r) := Q_p(d_p g^{-1} X_r, d_p g^{-1} Y_r)\,, \tag{2.4.5}$$

where $g(p) = q$. Notice that this is well defined since Q_p is K-invariant. In fact, if $gp = hp = q$, then $h^{-1}g \in K$, so that

$$\begin{aligned} Q_p(d_p g^{-1} X_r, d_p g^{-1} Y_r) &= Q_p(d_p(h^{-1}g)) d_p g^{-1} X_r, d_p(h^{-1}g)) d_p g^{-1} Y_r) \\ &= Q_p(d_p g^{-1} X_r, d_p g^{-1} Y_r)\,. \end{aligned}$$

This gives a G-invariant Riemannian metric on G/K and any other G-invariant Riemannian metric gives an inner product on $\mathfrak{p}$.

We need to define now the geodesic symmetries. We start with s_p. Once we'll have defined this, if $g(p) = q$ as above, then $s_p = g \circ s_p \circ g^{-1}$ will give the geodesic symmetry at any other point.

We define s_p as a map that satisfies the relation

$$s_p \circ \pi = \pi \circ \sigma \tag{2.4.6}$$

that is

$$s_p = \pi \circ \sigma \circ \pi^{-1}\,.$$

It is easy to see that s_p is well-defined. In fact, since $K \subset G^\sigma$, then

$$s_p(x) = \pi(\sigma(\pi^{-1}(x))) = \pi(\sigma(xK)) = \pi(\sigma(x)\sigma(K)) = \pi(\sigma(x))\,.$$

We see now that $s_p^2 = Id$. In fact, By applying once more s_p on the left of (2.4.6), we obtain

$$s_p \circ (s_p \circ \pi) = s_p \circ (\pi \circ \sigma) = (s_p \circ \pi) \circ \sigma = (\pi \circ \sigma) \circ \sigma = \pi \circ (\sigma)^2 = \pi\,,$$

so that $(s_p)^2 = Id$. The commutativity of the diagram

$$\begin{tikzcd} G \arrow[r, "\sigma"] \arrow[d, "\pi"'] & G \arrow[d, "\pi"] \\ G/K \arrow[r, "s_p"'] & G/K \end{tikzcd}$$

implies that also

$$\begin{CD} \mathfrak{p} @>{d_e\sigma}>> \mathfrak{p} \\ @V{d_e\pi}VV @VV{d_e\pi}V \\ T_p(G/K) @>>{d_ps_p}> T_p(G/K)\,, \end{CD}$$

commutes, so that if $X \in \mathfrak{p}$,

$$d_ps_p(d_e\pi(X)) = d_e\pi(d_e\sigma(X)) = -d_e\pi(X)\,,$$

that is $d_ps_p = -Id$. We will use this to verify that s_p is an isometry. Before doing this, we have however to gather some more information. Namely, from $s_p \circ \pi = \pi \circ \sigma$, we obtain that for $x \in G$

$$\begin{aligned} s_p \circ g(xK) &= s_p \circ \pi(gx) = \pi \circ \sigma(gx) \\ &= \sigma(gx)K = \sigma(g)\sigma(x)K \\ &= \sigma(g)(\pi \circ \sigma)(x) \\ &= \sigma(g)(s_p \circ \pi)(x) \\ &= (\sigma(g) \circ s_p)(xK) \\ &= \sigma(g) \circ s_p(xK)\,, \end{aligned}$$

that is

$$s_p \circ g = \sigma(g) \circ s_p\,. \tag{2.4.7}$$

In particular, since $s_p(p) = p$ and $g(p) = q$, it follows that $s_p(q) = \sigma(g)s_p(p) = \sigma(g)(p)$. Finally, we want to show that s_p is an isometry, that is that

$$Q_{s_p(x)}((d_xs_p)X_x, (d_xs_p)Y_x) = Q_p(X_p, Y_p)\,.$$

To this purpose we will use that if $r = gp$, then

$$s_p(r) = s_p(gp) = \sigma(g)(p)\,.$$

So, let X, Y be left invariant vector field on G/K, $(d_pg)X_p = X_r$ and $(d_pg)Y_p = Y_r$. Then

$$\begin{aligned} &Q_{s_p(r)}((d_rs_p)X_r, (d_rs_p)Y_r) \\ &= Q_{s_p(gp)}((d_{gp}s_p)(d_pg)X_p, (d_{gp}s_p)(d_pg)Y_p) \\ &= Q_{\sigma(g)(p)}((d_p(s_p \circ g)X_p, (d_p(s_p \circ g)Y_p) \\ &= Q_{\sigma(g)(p)}((d_p(\sigma(g) \circ s_p)X_p, (d_p(\sigma(g) \circ s_p)Y_p) \\ &= Q_{\sigma(g)(p)}(d_p\sigma(g)\underbrace{d_ps_pX_p}_{=-X_p}, d_p\sigma(g)\underbrace{d_ps_pY_p}_{=-Y_p}) \\ &= Q_{\sigma(g)(p)}(d_p\sigma(g)X_p, d_p\sigma(g)Y_p) \\ &= Q_p(X_p, Y_p) \\ &= Q_r(X_r, Y_r)\,, \end{aligned}$$

where in the first equality we used that the vector fields are invariant, in the second the chain rule, in the third (2.4.7), in the next the fact that $d_p s_p = -Id$, and in the last ones the definition of Q in (2.4.5) . Hence s_p is an isometry. □

The following proposition shows that, under very general conditions, the automorphism σ is completely determined by its set of fixed points G^σ. This will have important consequences, as the Definition 2.4.11 shows.

PROPOSITION 2.4.9. *Let (G, K) be a Riemannian symmetric pair, $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{z}$ the Lie algebra of the center of G. If $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, then there exists exactly one analytic involutive automorphism $\sigma \in \mathrm{Aut}(G)$, such that $(G^\sigma)^\circ \subset K \subset G^\sigma$.*

REMARK 2.4.10. Here are two cases in which the condition that $\mathfrak{k} \cap \mathfrak{z} = \{0\}$ is verified.

(1) If $\mathfrak{g}$ is semisimple then $\mathfrak{z} = \{0\}$, hence $\mathfrak{k} \cap \mathfrak{z} = \{0\}$.
(2) If (G, K) is a Riemannian symmetric pair associated to a globally symmetric space, then we proved in Theorem 2.2.7 that K contains no non-trivial subgroups normal in K, so that the Lie algebra of $\mathfrak{k}$ does not contain a subalgebra that is an ideal in $\mathfrak{g}$. Hence the involutive automorphism in Proposition 2.4.9 is exactly the one the Riemannian symmetric pair (G, K) came equipped with.

The uniqueness of the involutive automorphism of a Riemannian symmetric pair allows us to give the following definition: 2

DEFINITION 2.4.11. If (G, K) is a Riemannian symmetric pair with involution σ, the *Cartan involution* is defined as $\Theta := d_e\sigma$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to Θ.

To prove the proposition recall that the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$ is the trace form of $\mathrm{ad}_\mathfrak{g}$, that is, if $X, Y \in \mathfrak{g}$, then

$$B_\mathfrak{g}(X, Y) := \mathrm{tr}(\mathrm{ad}_\mathfrak{g}(X)\,\mathrm{ad}_\mathfrak{g}(Y)\,.$$

This is obviously a symmetric bilinear form and it is invariant under $\mathrm{Aut}(G)$, that is if $\tau \in \mathrm{Aut}(G)$, then

$$B_\mathfrak{g}(X, Y) = B_\mathfrak{g}(\tau_* X, \tau_* Y)\,.$$

To verify this observe that

$$\mathrm{ad}_\mathfrak{g}(\tau_* X)(Y) = [\tau_* X, Y] = \tau_*[X, \tau_*^{-1} Y] = \tau_*\,\mathrm{ad}_\mathfrak{g}(X)(\tau_*^{-1}(Y)) = (\tau_*\,\mathrm{ad}_\mathfrak{g}(X)\tau_*^{-1})(Y)$$

and hence

$$\begin{aligned} B_\mathfrak{g}(\tau_* X, \tau_* Y) &= \mathrm{tr}(\mathrm{ad}_\mathfrak{g}(\tau_* X)\,\mathrm{ad}_\mathfrak{g}(\tau_* Y)) = \mathrm{tr}(\tau_*\,\mathrm{ad}_\mathfrak{g}(X)\,\mathrm{ad}_\mathfrak{g}(Y)\tau_*^{-1}) \\ &= \mathrm{tr}(\mathrm{ad}_\mathfrak{g}(X)\,\mathrm{ad}_\mathfrak{g}(Y)) = B_\mathfrak{g}(X, Y)\,. \end{aligned}$$

Finally we will need the following:

LEMMA 2.4.12. *Let G be a connected Lie group, $K \leq G$ a compact subgroup and let $\mathfrak{z}$ the Lie algebra of the center of G. If $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, then $B_{\mathfrak{g}}$ is strictly negative definite.*

PROOF. Since K is compact, then $\mathrm{Ad}_{\mathrm{G}}(K)$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$ and hence there exists a strictly positive definite quadratic form q on $\mathfrak{g}$ such that $\mathrm{Ad}_{\mathrm{G}}(K) \leq \mathrm{O}(\mathfrak{g}, \mathrm{q})$. Thus $\mathrm{Ad}_{\mathrm{G}}(K)$ consists of matrices that are self-adjoint with respect to q and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ os skew-symmetric matrices, $\mathrm{ad}_{\mathfrak{g}}(X) = -\mathrm{ad}_{\mathfrak{g}}(X)^*$ for all $X \in \mathfrak{g}$. Thus for $x \in \mathfrak{k}$

$$B_{\mathfrak{g}}(X,X) = \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(X)) = \sum_{i,j=1}^{n} (\mathrm{ad}_{\mathfrak{g}}(X))_{i,j} (\mathrm{ad}_{\mathfrak{g}}(X))_{j,i} = -\sum_{j=1}^{n} (\mathrm{ad}_{\mathfrak{g}}(X))_{i,i}^2 \leq 0\,.$$

Thus $B_{\mathfrak{g}}(X,X) = 0$ if and only if $\mathrm{ad}_{\mathfrak{g}}(X))_{i,j} = 0$ for all i,j, that is $0 = \mathrm{ad}_{\mathfrak{g}}(X)(Y) = [X,Y]$ for all $X \in \mathfrak{k}$ and all $Y \in \mathfrak{g}$. Equivalently if and only if $X \in \mathfrak{z} \cap \mathfrak{k}$. $\square$

PROOF OF PROPOSITION 2.4.9. Let $\sigma_1, \sigma_2 \in \mathrm{Aut}(G)$ be two involutive automorphisms of G satisfying $(G^{\sigma_i})^\circ \subset K \subset G^{\sigma_i}$ for $i = 1,2$, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_i$ be the corresponding decomposition, where $\mathfrak{p}_i$ is the eigenspace corresponding to the eigenvalue -1 of $d_p\sigma_i$, $i = 1,2$. Since $\sigma_i \in \mathrm{Aut}(G)$, the Killing form $B_{\mathfrak{g}}$ is σ_i-invariant. It follows that $\mathfrak{k}$ is orthogonal to each of the $\mathfrak{p}_i$ with respect to $B_{\mathfrak{g}}$. In fact, if $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}_i$

$$B_{\mathfrak{g}}(X,Y) = B_{\mathfrak{g}}(d_o\sigma_i(X), d_o\sigma_i(Y)) = B_{\mathfrak{g}}(X,-Y)\,,$$

and hence $B_{\mathfrak{g}}(X,Y) = 0$.

Now if $Y_1 \in \mathfrak{p}_1$, and $Y_1 = X + Y_2$ is its orthogonal decomposition, then $X = Y_1 - Y_2 \in \mathfrak{k} \cap \mathfrak{k}^\perp$. Since $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, Lemma 2.4.12 implies that the Killing form on $\mathfrak{k}$ must be negative definite and hence $\mathfrak{k} \cap \mathfrak{k}^\perp = \{0\}$. It will then follow that $\mathfrak{p}_1 = \mathfrak{p}_2$ and $\sigma_1 = \sigma_2$. $\square$

We saw in Lemma 2.4.7 that such decomposition exists and now we know that it is unique.

LEMMA 2.4.13. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution Θ. Then*

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}\,, \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}\,.$$

PROOF. We prove that $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, the other inclusions are similar. Let $X \in \mathfrak{k}, Y \in \mathfrak{p}$. Then

$$\Theta[X,Y] = [\Theta X, \Theta Y] = [X,-Y] = -[X,Y]\,,$$

that is $[X,Y]$ belongs to the eigenspace of Θ with eigenvalue -1. $\square$

2.5. Exponential Maps and Geodesics

Let (G, K) be a Riemannian symmetric pair associated to a a Riemannian symmetric space M with base point $p \in M$. By the Remark 2.4.10(2), there is a unique involution σ and hence the Cartan decomposition of $\mathfrak{g}$ is unique. Let $\pi : G \to M$ be the projection map $g \mapsto g(p)$, let $\exp : \mathfrak{g} \to G$ be the Lie group exponential map and $\mathrm{Exp}_p : T_pM \to M$ the Riemannian exponential map.

The following theorem gives the relation between the two exponential maps, namely:

THEOREM 2.5.1. *The following diagram*

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "d_e\pi"] \arrow[d, "\exp"'] & T_pM \arrow[d, "\mathrm{Exp}_p"] \\ G \arrow[r, "\pi"'] & M \end{tikzcd}$$

commutes, that is $\pi(\exp(X)) = \mathrm{Exp}_p(d_e\pi(X))$ *for any* $X \in \mathfrak{p}$.

The geodesic $\gamma : \mathbb{R} \to M$ *with* $\gamma(0) = p \in M$ *and with tangent vector* $d_e\pi(X)$, *for* $X \in \mathfrak{p}$ *is given by*

$$\gamma_{d_e\pi(X)}(t) = \exp(tX)(p)\,.$$

Moreover every geodesic through any point $q \in M$ *is of this form.*

We start the proof with the following:

LEMMA 2.5.2. *Let* M *be a Riemannian symmetric space and* $\gamma : \mathbb{R} \to M$ *a geodesic. Then for every* $t, \ell \in \mathbb{R}$

$$s_{\gamma(t+\ell)} = s_{\gamma(t)} s_{\gamma(0)} s_{\gamma(\ell)} \tag{2.5.1}$$

and the map $t \mapsto \mathcal{T}_t$ *that to* $t \in \mathbb{R}$ *associates the translation by* t *along the geodesic* γ *implemented by the transvection* $\mathcal{T}_t$ *is a homomorphism.*

PROOF OF THEOREM 2.5.1. We saw that for every $a, b \in \mathbb{R}$, $s_{\gamma(a)}\gamma(b) = \gamma(2a - b)$, from which it follows that

$$s_{\gamma(t_1)} s_{\gamma(t_2)} = \mathcal{T}_{2(t_1-t_2)}\,.$$

Thus the translation depends only on the difference $t_1 - t_2$ and hence

$$s_{\gamma(t+\ell)} s_{\gamma(\ell)} = s_{\gamma(t)} s_{\gamma(0)}\,,$$

which, composing on the right by $s_{\gamma(\ell)}$ gives (2.5.1). Now, using (2.5.1) with $\ell = t'/2$, we have

$$\mathcal{T}_{t+t'} = s_{\gamma(t/2+t'/2)} s_{\gamma(0)} = s_{\gamma(t/2)} s_{\gamma(0)} s_{\gamma(t'/2)} s_{\gamma(0)} = \mathcal{T}_t \mathcal{T}_{t'}\,.$$

□

PROOF. Let $X \in \mathfrak{p}$ and let $\gamma : \mathbb{R} \to M$ be the geodesic in M such that $\gamma(0) = o$ and $\dot\gamma(0) = d_e\pi X$ (hence $\gamma(t) = \gamma_{d_e\pi(X)}(t)$ in the previous notation). Let $\mathcal{T}_t$ be the translation by t along γ. Since $\mathcal{T}_t$ by the previous lemma is a homomorphim, it is a one-parameter subgroup of G and hence there exists $Y \in \mathfrak{g}$ such that $\mathcal{T}_t = \exp(tY)$ for all $t \in \mathbb{R}$.

We are going to show that $Y \in \mathfrak{p}$, that is that $d_e\sigma(Y) = -Y$. To this purpose, observe that because of (2.5.1) and because of the definition of σ, $\sigma(g) = s_{\gamma(0)} g s_{\gamma(0)}$, we have

$$\sigma(\mathcal{T}_t) = \sigma(s_{\gamma(t/2)} s_{\gamma(0)}) = \sigma(s_{\gamma(0)} s_{\gamma(-t/2)}) = s_{\gamma(-t/2)} s_{\gamma(0)} = \mathcal{T}_{-t}\,,$$

or $\sigma(\exp tY) = \exp(-tY)$. By differentiating this equality we obtain

$$(d_e\sigma)Y = \frac{d}{dt}\bigg|_{t=0} \sigma(\exp tY) = \frac{d}{dt}\bigg|_{t=0} \exp(-tY) = -Y\,,$$

so that $Y \in \mathfrak{p}$.

To conclude that $X = Y$, notice that by definition $\pi(\mathcal{T}_t) = \mathcal{T}_t(p) = \gamma(t)$. By differentiating this expression, we obtain that

$$\frac{d}{dt}\bigg|_{t=0} \pi(\mathcal{T}_t) = \frac{d}{dt}\bigg|_{t=0} \gamma(t) = (d_e\pi)X\,.$$

On the other hand by the chain rule

$$\frac{d}{dt}\bigg|_{t=0} \pi(\mathcal{T}_t) = d_e\pi\, \frac{d}{dt}\bigg|_{t=0} \mathcal{T}_t = (d_e\pi)Y\,,$$

and thus $X = Y$.

Finally if now η is any geodesic through a generic point $q \in M$, there exists $g \in G$, such that $\eta(0) = g \cdot p$. Hence $g^{-1}\eta$ is a geodesic which goes through p when $t = 0$, and is hence of the form $g^{-1}\eta(t) = \exp(tX)(p)$. Thus every geodesic η, is of the form $\eta(t) = g\exp(tX)(p)$ for some $X \in \mathfrak{p}$. $\square$

The above theorem shows, in particular, that the Riemannian exponential map $\mathrm{Exp} : TM \to M$ does not depend on its Riemannian metric and gives a formula for the geodesics in M.

We are now interested in finding a formula for the derivative of the Riemannian exponential map at a point $X \in \mathfrak{p}$, formula that we will use both in computing the curvature tensor in § 2.6 and in characterizing the totally geodesic submanifolds of a Riemannian symmetric space in § 2.7. This will go in three steps:

(1) We will use, without a proof, the formula in Theorem A.2.10 for the differential at a point $X \in T_pM$ of the exponential map Exp_∇ associated to an (analytic) connection ∇.
(2) We will construct in Lemma 2.5.4 a connection on a Lie group G whose geodesics through $e \in G$ are the one-parameter subgroups of G and thus show that $\mathrm{Exp}_\nabla = \exp$. Then with the use of Theorem A.2.10 we can

deduce in Theorem 2.5.3 a formula for the differential of the Lie group exponential map.
(3) Finally in Corollary 2.5.5 we will use the formula for the differential of the Lie group exponential in Theorem 2.5.1 and the relation in Theorem 2.5.1 of the Lie group exponential with the Riemannian exponential on G/K to deduce the needed formula for the differential of the Riemannian exponential map Exp on $M = G/K$.

THEOREM 2.5.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let $\exp : \mathfrak{g} \to G$ the Lie group exponential map. By identifying $T_X\mathfrak{g} \cong \mathfrak{g}$, so that $d_X \exp : T_X\mathfrak{g} \cong \mathfrak{g} \to T_{\exp(X)}G$, we have*

$$d_X \exp = d_e L_{\exp X} \circ \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{\mathfrak{g}}^n X)}{(n+1)!} \tag{2.5.2}$$

As announced, we will need the following lemma, whose proof we postpone to the end of the section.

LEMMA 2.5.4. *There exists a connection ∇ on G such that:*
(1) *∇ is G-invariant (in the sense of* (A.2.1)*);*
(2) *$\nabla_{\widetilde{X}}\widetilde{Y} = 0$ for every left invariant vector fields $\widetilde{X}, \widetilde{Y} \in \mathrm{Vect}^G(G)$;*
(3) *the exponential map associated to ∇ coincides with the Lie group exponential map $\mathrm{Exp}_\nabla = \exp$.*

PROOF OF THEOREM 2.5.3. Because of Lemma 2.5.4(2), left invariant vector fields are parallel, hence on a normal neighborhood $\widetilde{X} = X^*$ by uniqueness. Let $\widetilde{X}_e = X \in \mathfrak{g}$ and $\widetilde{Y}_e = Y \in \mathfrak{g}$. Then by Theorem A.2.10 applied to $M = G$ and $q = e \in G$ we have that for $X, Y \in \mathfrak{g}$,

$$(d_X(\mathrm{Exp}_\nabla))(Y) = \left(\sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(\widetilde{X}))^n}{(n+1)!}(\widetilde{Y}) \right)_{(\mathrm{Exp}_\nabla)(X)} . \tag{2.5.3}$$

Since the vector field $\sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(\widetilde{X}))^n}{(n+1)!}(\widetilde{Y})$ is left invariant, then

$$\left(\sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(\widetilde{X}))^n}{(n+1)!}(\widetilde{Y}) \right)_{(\mathrm{Exp}_\nabla)(X)} = d_e L_{(\mathrm{Exp}_\nabla)(X)} \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(X))^n}{(n+1)!}(Y) ,$$

so that (2.5.3) becomes

$$(d_X(\mathrm{Exp}_\nabla))(Y) = d_e L_{(\mathrm{Exp}_\nabla)(X)} \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(X))^n}{(n+1)!}(Y) . \tag{2.5.4}$$

With the use of and Lemma 2.5.4(3) this formula is exactly the one we needed to show. $\square$

Before getting to the proof of the Lemma we deduce the formula we will need. Let $M = G/K$ be a symmetric space, $p \in M$ a basepoint with $K = \mathrm{Stab}_G(p)$ and $\pi : G \to G/K$. Recall that $d_e\pi : \mathfrak{p} \to T_p(G/K)$ is an isomorphism and we can define $\mathrm{Exp} \circ d_e\pi \colon \mathfrak{p} \to T_p(G/K) \to G/K$. Then we have:

Corollary 2.5.5. *The differential*

$$d_X(\mathrm{Exp} \circ d_e\pi) \colon \mathfrak{p} \to T_{(\mathrm{Exp}(X) \circ d_e(X))(p)} M$$

of the Riemannian exponential map

$$\mathrm{Exp} \circ d_e\pi \colon \mathfrak{p} \to G/K$$

is given by

$$d_X(\mathrm{Exp} \circ d_e\pi) = d_p L_{\exp X} \circ \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!}\,, \tag{2.5.5}$$

where $T_X = (\mathrm{ad}_{\mathfrak{g}} X)^2$ *for* $X \in \mathfrak{p}$.

Proof. We recall that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/K \\ {\scriptstyle L_g}\downarrow & & \downarrow{\scriptstyle L_g} \\ G & \xrightarrow[\pi]{} & G/K \end{array}$$

commutes, so that

$$\pi \circ L_{\exp X} = L_{\exp X} \circ \pi\,. \tag{2.5.6}$$

In Theorem 2.5.1 have proven that for any $X \in \mathfrak{p}$, $\pi \circ \exp(X)|_{\mathfrak{p}} = \mathrm{Exp}_e \circ d_e\pi X|_{\mathfrak{p}}$, so that, if we set $\mathcal{L}(X) := \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}} X)^n}{(n+1)!}$,

$$\begin{aligned} d_X(\mathrm{Exp}_e \circ d_e\pi X|_{\mathfrak{p}}) =& d_X(\pi \circ \exp(X)|_{\mathfrak{p}}) \\ =& d_{\exp X}\pi \circ d_X(\exp|_{\mathfrak{p}}) \\ =& d_{\exp X}\pi \circ d_X(\exp)|_{\mathfrak{p}} \\ =& d_{\exp X}\pi \circ d_e L_{\exp X} \circ \mathcal{L}(X) \\ =& d_e(\pi \circ L_{\exp X}) \circ \mathcal{L}(X)|_{\mathfrak{p}} \\ =& d_e(L_{\exp X} \circ \pi) \circ \mathcal{L}(X)|_{\mathfrak{p}} \\ =& (d_p L_{\exp X}) \circ d_e\pi \circ \mathcal{L}(X)|_{\mathfrak{p}} \end{aligned}$$

where we used in the fourth inequality Theorem 2.5.3 and in the sixth one (2.5.6).

Now observe that, because of Lemma 2.4.13, if $Y \in \mathfrak{p}$,

$$\mathrm{ad}_{\mathfrak{g}}(X)^n(Y) \in \begin{cases} \mathfrak{k} & \text{if } n \text{ is odd} \\ \mathfrak{p} & \text{if } n \text{ is even,} \end{cases}$$

so that

$$d_e\pi \circ \mathrm{ad}_{\mathfrak{g}}(X)^n(Y) \begin{cases} = 0 & \text{if } n \text{ is odd} \\ = \mathrm{ad}_{\mathfrak{g}}(X)^n(Y) & \text{if } n \text{ is even.} \end{cases}$$

Thus

$$d_e\pi \circ \mathcal{L}(X)|_{\mathfrak{p}} = d_e \circ \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}} X)^n}{(n+1)!}\Bigg|_{\mathfrak{p}} = \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_{\mathfrak{g}} X)^{2n}}{(2n+1)!}\,,$$

which completes the proof. □

We are now left with the proof of the lemma.

Sketch of the proof of Lemma 2.5.4. Let $C^\infty(G,\mathfrak{g})$ bethe space of C^∞ maps $G \to \mathfrak{g}$ and define

$$\begin{aligned} X\colon C^\infty(G,\mathfrak{g}) &\longrightarrow \quad \mathrm{Vect}(G) \\ F \quad &\longmapsto \{g \mapsto d_e L_g F(g)\} \end{aligned}$$

which is an isomorphism of C^∞-modules. It is not difficult to verify that there is a correspondence between connections on G and maps

$$D\colon C^\infty(G,\mathfrak{g}) \times C^\infty(G,\mathfrak{g}) \to C^\infty(G,\mathfrak{g})$$

that satisfy the following properties:

(1) D is $\mathbb{R}$-linear in the first variable and C^∞-linear in the second variable;
(2) (Leibniz rule) for every $f \in C^\infty(G)$, $F_1, F_2 \in C^\infty(G,\mathfrak{g})$,

$$D_{F_1}(fF_2)(g) = f(g)D_{F_1}F_2(g) + d_e(f\circ L_g)(F_1(g))F_2(g)\,.$$

(Notice that here $(f\circ L_g)(h) = f(hg)$ and $d_e(f\circ L_g)\colon T_eG \to R$ is hence a linear form whose evaluation at $F_1(g)$ fives a function of g that then multiplies $F_2(g)$.)

In fact for every connection $\nabla\colon\ \mathrm{Vect}(G)\times\mathrm{Vect}(G) \to \mathrm{Vect}(G)$ one can define a map $D\colon C^\infty(G,\mathfrak{g}) \times C^\infty(G,\mathfrak{g}) \to C^\infty(G,\mathfrak{g})$ by

$$(D_{F_1}F_2)(g) := (d_eL_g)^{-1}((\nabla_{X(F_1)}X(F_2))(g))\,,$$

and likewise every map D that satisfies (1) and (2) gives rise to a connection ∇ on G via

$$\nabla_{X(F_1)}X(F_2)(g) := d_eL_g(D_{F_1}F_2)(g)\,,$$

so that the diagram

$$\begin{tikzcd} C^\infty(G,\mathfrak{g}) \times C^\infty(G,\mathfrak{g}) \arrow[r, "D"] \arrow[d, "X\times X"'] & C^\infty(G,\mathfrak{g}) \arrow[d, "X"] \\ \mathrm{Vect}(G)\times\mathrm{Vect}(G) \arrow[r, "\nabla"'] & \mathrm{Vect}^G(G) \end{tikzcd}$$

commutes.

Now in order to define a connection ∇ we define

$$(D_{F_1}F_2)(g) := \frac{d}{dt}F_2(g\exp tF_1(g))\bigg|_{t=0} = d_e(F_2\circ L_g)\colon T_eG = \mathfrak{g} \to T_{F_2(g)}\mathfrak{g} \cong \mathfrak{g}\,.$$

One can verify that D satisfies (1) and (2) and moreover it is G-invariant.

Finally, observe that $X(F)$ is left invariant if and only if F is constant. In fact $X(F)$ is left invariant if and only if $X(F)_g = (d_eL_{gh^{-1}})X(F)_h$ for every $g,h\in G$. Since

$$(d_eL_{gh^{-1}})X(F)_h = (d_eL_{gh^{-1}})(d_eL_h)F(h) = (d_eL_g)F(h)$$

and

$$X(F)_g = (d_eL_g)F(g)\,,$$

left invariance is equivalent to $F(g) = F(h)$ for every $g,h\in G$. Thus if F_1,F_2 are constant, then $D_{F_1}F_2 = 0$.

The only thing left to verify is that the Lie group exponential and the exponential coming from the left invariant connection are the same. To this purpose, let X be a left invariant vector field and let γ_X be the unique integral curve such that $\gamma_X(0) = e$ and $\dot\gamma_X(t) = X_t$. (It is not difficult to show that although the existence of the integral cure is usually only local, γ_X can be defined for all $t\in\mathbb{R}$.) It will be enough to show that this integral curve is a geodesic and a homomorphism, because in this case it will be both the geodesic that defines the exponential coming from the left invariant connection and one-parameter subgroup that defines the Lie group exponential. Since $\nabla_XY = 0$, then $\nabla_{\dot\gamma_X(t)}\dot\gamma_Y(t) = 0$ and hence γ_X is a geodesic. To see that it is a homomorphism, since ∇ is left invariant, for all $s\in\mathbb{R}$ the two curves $t\mapsto\gamma_X(t+s)$ and $t\mapsto\gamma_X(s)\gamma_X(t)$ are both geodesics in G and both go through the point $\gamma_X(s)$ when $t=0$. Moreover

$$\frac{d}{dt}\bigg|_{t=0}(\gamma_X(s)\gamma_X(t)) = d_{\gamma_X(t)}L_{\gamma_X(s)}\dot\gamma_X(0) = \dot\gamma_X(s)\,,$$

where the last equality follow from the definition of γ_X and the fact that X is left invariant. Hence $\gamma_X(s+t) = \gamma_X(t)\gamma_X(s)$ and the proof is complete. $\square$

2.6. Curvature

The goal of this section is to prove the following

THEOREM 2.6.1. *Let (G,K) be a symmetric pair as in § 2.5 and let R be the curvature tensor of G/K with respect to the Riemannian metric Q. Then at the point $o\in G/K$*

$$R_o(\bar X_1,\bar X_2)\bar X_3 = -[[\bar X_1,\bar X_2],\bar X_3]$$

where $\bar X_i = d_e\pi X_i$, for $X_i\in\mathfrak{p}$, $i=1,2,3$.

Sketch of the proof. We compute first the sectional curvature, then use it to compute the curvature tensor. The proof follows [**Hel01**, Theorem IV.4.2, p. 215].

To compute the sectional curvature we use the formula in [**Hel01**, Lemma I.12.1, p.64], according to which the sectional curvature can be computed as

$$K = -\frac{3}{2}\Delta(f)(0)\,,$$

where f is the function giving the ratio of the surface elements in the tangent plane and the surface, and where Δ is the Laplacian, that is the operator $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, with respect to the coordinate functions x_1, x_2 in an orthonormal basis.

In our specific case, let $\{X_1, \dots, X_n\}$ be an orthonormal basis of $\mathfrak{p}$ and $P \subset \mathfrak{p}$ be a two-dimensional subspace spanned by $\{X_1, X_2\}$. Let $N_0 \subset \mathfrak{p}$ be a normal neighborhood of the origin and give $\Sigma := \operatorname{Exp}(P \cap N_0) \subset G/K$ the induced metric from $M = G/K$. Then the exponential maps of Σ and of M coincide when restricted to $P \cap N_0$, and hence we can use (2.5.5) to compute $d_X \operatorname{Exp}$. In other words, if $X \in P \cap N_0$,

$$f(X) = \frac{|d_X \operatorname{Exp} X_1 \vee d_X \operatorname{Exp} X_2|}{|X_1 \vee X_2|} = \sum_{n=0}^{\infty} \frac{(T_X)^n(X_1)}{(2n+1)!} \vee \sum_{n=0}^{\infty} \frac{(T_X)^n(X_2)}{(2n+1)!}\,,$$

since the X_1, X_2 are orthonormal and left translations are isometries. Writing $X = x_1X_1 + x_2X_2$, expressing T_X in terms of x_1 and x_2 and computing the derivatives in the Laplacian, with some patience one obtains that

$$K(P) = Q_0(\operatorname{ad}_{\mathfrak{g}}([X_1, X_2])X_1, X_2)\,. \tag{2.6.1}$$

Define now a quadrilinear form

$$B(X, Y, Z, T) := Q_0((R(X, Y) + \operatorname{ad}_{\mathfrak{g}}[X, Y])Z, T)\,,$$

for $X, Y, Z, T \in \mathfrak{p}$. Because of (2.6.1) and of the definition of sectional curvature in (A.2.2), then $B(X_1, X_2, X_1, X_2) = 0$. From this and from the properties of the curvature, it follows that $B(X, Y, X, Y) = 0$ for all $X, Y \in \mathfrak{p}$. In fact from (R_1) and (R_2) in (A.2.3) if

$$X = x_1X_1 + x_2X_2 \qquad \text{and} \qquad Y = y_1X_1 + y_2Y_2\,,$$

then

$$B(X, Y, X, Y) = (x_1y_2 - x_2y_1)^2 B(X_1, X_2, X_1, X_2)\,,$$

so that $B(X, Y, X, Y) = 0$ for all $X, Y \in \mathfrak{p}$. Then playing around with the symmetries of B (see for example [**Hel01**, Lemma I.12.4, p. 68]), it is easy to see that $B \equiv 0$. □

Corollary 2.6.2. *The Levi-Civita connection of a Riemannian globally symmetric space is independent of the metric.*

2.7. Totally Geodesic Submanifolds

DEFINITION 2.7.1. A subspace $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is a *Lie triple system* if $[[X,Y],Z] \in \mathfrak{n}$ for all $X, Y, Z \in \mathfrak{n}$.

Lie triple systems correspond to totally geodesic submanifolds in the following sense:

THEOREM 2.7.2. *Let M be a Riemannian globally symmetric space and let $\mathfrak{n}$ be a Lie triple system in $\mathfrak{p}$. Assume that $M = G/K$, where $K = \mathrm{Stab}_G(p_0)$. Then $N := \operatorname{Exp}\mathfrak{n}$ is a totally geodesic submanifold of M such that $T_{p_0}N = \mathfrak{n}$.*

Conversely, if N is a totally geodesic submanifold and $p_0 \in N$, then the subspace $\mathfrak{n} = T_{p_0}N$ is a Lie triple system.

PROOF. Let $\mathfrak{n} \subset \mathfrak{p}$ be a Lie triple system. Then $[\mathfrak{n},\mathfrak{n}] \subset [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. We verify that $\mathfrak{n} + [\mathfrak{n},\mathfrak{n}] \subset \mathfrak{g}$ is a Lie subalgebra. To see this, we remark first of all that since $\mathfrak{n}$ is a Lie triple system, then $[\mathfrak{n},\mathfrak{n}]$ is a Lie subalgebra. In fact, if $X, Y, Z, W \in \mathfrak{n}$, then, the Jacobi identity applied to $[X,Y]$, Z and W reads

$$0 = [[X,Y],[Z,W]] + [[Y,[Z,W]],X] + [[[Z,W],X],Y]\,,$$

where $[Y,[Z,W]], [[Z,W],X] \in \mathfrak{n}$. Hence

$$[[X,Y],[Z,W]] = -[[Y,[Z,W]],X] - [[[Z,W],X],Y] \in [\mathfrak{n},\mathfrak{n}]\,.$$

It follows that if $X, Y, Z, X', Y', Z' \in \mathfrak{n}$, then

$$[X + [Y,Z], X' + [Y',Z']] \in \mathfrak{n} + [\mathfrak{n},\mathfrak{n}] =: \mathfrak{g}'\,.$$

Let now $G' < G$ be the analytic subgroup whose Lie algebra is $\mathfrak{g}'$. Let $M' := G' \cdot p_0$ and let $K' = \mathrm{Stab}_{G'}(p_0)$. Then $K' < G'$ is closed (since the G'-action on M is continuous) and hence G'/K' has a differentiable structure that can be induced on $M' \cong G'/K'$. It follows that M' is a submanifold of M and $T_{p_0}M' \cong \mathfrak{n}$. (In fact, it is enough to see that $\mathrm{Lie}(K') = [\mathfrak{n},\mathfrak{n}]$. But $\mathrm{Lie}(K') \subset \mathrm{Lie}(K) \cup \mathrm{Lie}(G') = \mathfrak{k} \cap [\mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$, so that if $X = X_1 + X_2 \in \mathrm{Lie}(K')$, then $X = X_2$.) The M-geodesic through p_0 are all of the form $\exp(tX) \cdot p_0$ for $X \in \mathfrak{p}$ and $X \in \mathfrak{n}$ if and only if $\exp(tX) \cdot p_0 \in M'$. It follows that M' is geodesic at p_0 and, since G' is transitive, it is totally geodesic and $M' = \operatorname{Exp}\mathfrak{n}$ (since $\operatorname{Exp}_{p_0}(tX) = \exp_G(tX)p_0 = \exp_{G'}(tX)p_0$.)

To see the converse, let $N \subset M$ be a totally geodesic submanifold. Then for $X, Y \in T_{p_0}N$ the geodesics $t \mapsto \operatorname{Exp}(tX)$ and $t \mapsto \operatorname{Exp}(tY)$ are in N, so that we can consider the restriction $\operatorname{Exp} : T_{p_0}N \to N$ and its differential $d_{tY}\operatorname{Exp} : T_{p_0}N \to T_{\operatorname{Exp}(tY)}N$.

It follows from Corollary 2.5.5, we have that

$$d_{tY}\operatorname{Exp}(X) = d_o L_{\exp tY} \circ \sum_{n=0}^{\infty} \frac{(T_{tY})^n}{(2n+1)!}(X)\,.$$

By definition, the vector $d_{tY}\operatorname{Exp}(X)$ is tangent to the geodesic $t \mapsto \operatorname{Exp}(tY)$ so that, by parallel translation, $d_o L_{\exp(-tY)} \circ d_{tY}\operatorname{Exp}(X)$ is parallel to $d_{tY}\operatorname{Exp}(X)$

along $\exp(tY)$ and hence it is in $T_{p_0}N =:_{\mathfrak{n}}$. But for all $t \in \mathbb{R}$

$$d_o L_{\exp(-tY)} \circ d_{tY} \operatorname{Exp}(X) = \sum_{n=0}^{\infty} \frac{(T_{tY})^n}{(2n+1)!}(X) \in \mathfrak{n}\,,$$

so that $T_Y(X) \in \mathfrak{n}$. But

$$\begin{aligned} T_{Y+Z}(X) =& \operatorname{ad}_{\mathfrak{g}}(Y+Z)(\operatorname{ad}_{\mathfrak{g}}(Y+Z)(X)) \\ =&[Y+Z,[Y+Z,X]] \\ =&[Y+Z,[Y,X]+[Z,X]] \\ =&[Y,[Y,X]]+[Y,[Z,X]]+[Z,[Y,X]]+[Z,[Z,X]] \\ =&T_Y(X)+T_Z(X)+[Y,[Z,X]]+[Z,[Y,X]]\,, \end{aligned}$$

so that

$$[Y,[Z,X]]+[Z,[Y,X]] = T_{Y+Z}(X)-T_Y(X)-T_Z(X) \in \mathfrak{n}\,.$$

By the Jacobi identity

$$\begin{aligned} \mathfrak{n} \ni [Y,[Z,X]]+[Z,[Y,X]] =&[Y,[Z,X]]+[[Z,Y],X]+[Y,[Z,X]] \\ =&2[Y,[Z,X]]+[[Z,Y,X]] \\ =&2[Y,[Z,X]]+[X,[Y,Z]] \end{aligned} \tag{2.7.1}$$

and, exchanging the roles of X and Y,

$$2[X,[Z,Y]]+[Y,[X,Z]] \in \mathfrak{n}\,. \tag{2.7.2}$$

Hence it follows from (2.7.1) and (2.7.2), and using twice the Jacobi identity, that

$$\begin{aligned} \mathfrak{n} \ni& 2[Y,[Z,X]]+[X,[Y,Z]]-(2[X,[Z,Y]]+[Y,[X,Z]]) \\ =&2[X,[Z,Y]]+2[Z,[Y,X]]+[X,[Y,Z]]-(2[X,[Z,Y]]+[Y,[X,Z]]) \\ =&3[Y,[Z,X]]+3[X,[Y,Z]] \\ =&3[Z,[Y,X]]\,, \end{aligned}$$

that is $\mathfrak{n}$ is a Lie triple system. □

2.8. Examples

PROBABLY GET RID OF WHAT IS TYPES AND COPY FROM THE HANDWRITTEN NOTES

2.8.1. $\mathrm{SL(n,\mathbb{R})/SO(n)}$**.** Let us consider

$$G = \mathrm{SL(n,\mathbb{R})} = \{\mathrm{g} \in \mathrm{M_{t\times n}(\mathbb{R})} :\, \det \mathrm{g} = 1\}\,.$$

We consider the involutive automorphism $\sigma : G \to G$, define dby $p \mapsto (g^t)^{-1}$. Then

$$G^s igma = \{g \in G :\, (g^t)^{-1} = g\} = \{g \in G : g^t g = e\} = \mathrm{SO(n)} =: \mathrm{K}\,.$$

The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) = \mathfrak{sl}(n, \mathbb{R})$ consists of all $(n \times n)$-matrices with trace 0 and entries in $\mathbb{R}$ and the Cartan involution $\Theta : \mathfrak{sl}(n, \mathbb{R}) \to \mathfrak{sl}(n, \mathbb{R})$ is given by $\Theta(X) := -X^t$ for $X \in \mathfrak{sl}(n, \mathbb{R})$. The Cartan decomposition is hence

$$X = \frac{1}{2}(X - X^t) + +\frac{1}{2}(X + X^t)\,,$$

that is the decomposition of X into its antisymmetric and symmetric part. If $\mathfrak{so}(n)$ denotes the Lie algebra of SO(n) and $\mathrm{sym}_0(\mathrm{n})$ the set of symmetric $(n \times n)$-matrices of trace zero with entries in $\mathbb{R}$, we have

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{sym}_0(\mathrm{n})\,.$$

Denote $o = eK \in G/K$ the base point and consider the positive definite symmetric bilinear form

$$< X, Y >:= \mathrm{tr}(XY)\,,$$

where $X, Y \in T_o(G/K) \cong \mathfrak{p} \subset \mathfrak{g}$. This scalar product on $T_o(G/K)$ can be extended by left G-invariance to a G-invariant Riemannian metric on G/K.

The set $\mathrm{Pos}_1(n)$ of positive definite symmetric $(n \times n)$-matrices with determinant one and real entries can be identified with G/K as follows: from elementary linear algebra we know that any matrix $p \in \mathrm{Pos}_1(n)$ can be written as a matrix product $p = b^t b$, where $b \in \mathrm{SL}(\mathrm{n}, \mathbb{R})$. The group $\mathrm{SL}(\mathrm{n}, \mathbb{R})$ acts transitively on $\mathrm{Pos}_1(n)$ via

$$g \cdot p := g^t p g\,,$$

where $p \in \mathrm{Pos}_1(n)$ and $g \in \mathrm{SL}(\mathrm{n}, \mathbb{R})$. If $I_n \in \mathrm{Pos}_1(n)$ is the identity matrix, then $\mathrm{SO(n)} = \mathrm{Stab}_{\mathrm{SL(n,\mathbb{R})}}(\mathrm{I_n})$.

If $n = 2$, we can identify G/K (endowed with the G-invariant Riemannian metric induced as above) with the real hyperbolic plane $\mathcal{H}^2 := \{x + iy : x \in]RR, y > 0\}$ endowed with the metric $ds^2 = (dx^2 + dy^2)/y^2$. Indeed $\mathrm{SL}(2, \mathbb{R})$ acts transitively by isometries via linea transformations on $\mathcal{H}^2$ and SO(2) is the isotropy subgroup of $i \in \mathcal{H}^2$. So $\mathrm{Pos}_1(2)$ with a metric rescaled by a factor of 2 can be identified with the hyperbolic plane $(\mathcal{H}^2, ds^2)$.

2.8.2. G/K, where $G < \mathrm{SL}(\mathrm{n}, \mathbb{R})$ is a closed adjoint subgroup (i.e. $G^t = G$). As an involutive automorphism we take again $\sigma(g) := (g^t)^{-1}$, so that $K = G \cap \mathrm{SO(n)}$. if $o = eK \in G/K$ denotes the base point, endow G/K with a left invariant Riemannian metric as above.

2.8.2.1. The group $G = \mathrm{SO(p, q)}$ of linear transformations leaving invariant the bilinear form

$$Q(x, y) = -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{p+q} x_j y_j$$

on $\mathbb{R}^{p+q}$ is invariant under transposition. So if $K = G \cap \mathrm{SO(p+q)} = \mathrm{SO(p)} \times \mathrm{SO(q)}$, we get a symmetric space G/K.

The Cartan decomposition of the Lie algebra of SO(p, q) is given by $\mathfrak{so}(p,q) = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{s}(p) \times \mathfrak{s}(q) \subset \mathfrak{sp}(p+q)$ is the Lie algebra of K and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ B^t & o \end{pmatrix} : B \in M_{p\times q}(\mathbb{R}) \right\} \subset \mathrm{sym}_0(\mathrm{p}+\mathrm{q})\,.$$

In particular if $p = 1$, this symmetric space, with an appropriately rescaled metric, is isometric to the q-dimensional hyperbolic space.

2.8.2.2. The group $G = \mathrm{Sp}(2\mathrm{q}, \mathbb{R})$ leaving invariant the standard symplectic form

$$\omega(x,y) = \sum_{i=1}^{q} x_i y_{q+i} - \sum_{j=i}^{q} x_{q+j} y_j$$

on $\mathbb{R}^{2q}$ is invariant under transposition. If $K = \mathrm{Sp}(2\mathrm{q}, \mathbb{R}) \cap \mathrm{SO}(2\mathrm{q})$, then G/K is a symmetric space.

The Cartan decomposition of Lie(G) is given by of $\mathfrak{sp}(2q, \mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B^t & a \end{pmatrix} : A, B \in M_{q\times q}(\mathbb{R}),\ A^t = -A \right\} \subset \mathfrak{so}(2q)$$

is the Lie algebra of K and

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ -B^t & a \end{pmatrix} : A, B \in M_{q\times q}(\mathbb{R}),\ A^t = A \right\} \subset \mathrm{sym}_0(2\mathrm{q})\,.$$

Recall that a complex structure on a real vector space V is an endomorphism J of V with the property that $J^2 = -Id_V$. Moreover if $g \in \mathrm{GL}(V)$, then $g \circ J \circ g^{-1}$ is also a complex structure.

Consider the set S_{2q} of complex structures on the symplectic vector space $(\mathbb{R}^{2q}, \omega)$ such that the symmetric bilinear from defined on $\mathbb{R}^{2q}$ by

$$q_J(x,y) := \omega(x, Jy)$$

is positive definite. A complex structure with this property is called *ω-compatible.* The group $G = \mathrm{Sp}(2\mathrm{q}, \mathbb{R})$ acts naturally on S_{2q} by conjugation, namely $g \cdot J = gJg^{-1}$, for $g \in \mathrm{Sp}(2\mathrm{q}, \mathbb{R})$ and $J \in S_{2q}$. Indeed if $g \in \mathrm{Sp}(2\mathrm{q}, \mathbb{R})$, then

$$q_{g\cdot J}(x,y) = \omega(x, gJg^{-1}y) = \omega(g^{-1}x, Jg^{-1}y) = q_J(g^{-1}x, g^{-1}y)\,,$$

so that $q_{g\cdot J}$ is positive definite if q_J is. Moreover the action is transitive. Indeed we choose as a base point $o \in S^{2q}$ the ω-compatible complex structure given by the matrix

$$J_0 := \begin{pmatrix} 0 & -Id_q \\ Id_q & 0 \end{pmatrix}\;;$$

its associates symmetric bilinear form q_{J_0} is the standard scalar product in $\mathbb{R}^{2q}$. then the isotropy subgroup of G at o is precisely the group $K = \mathrm{Sp}(2\mathrm{q}, \mathbb{R}) \cap \mathrm{SO}(2\mathrm{q})$, so $S_{2q} = \mathrm{Sp}(2\mathrm{q}, \mathbb{R}) \cdot \mathrm{o}$ can be identified with G/K.

If $q = 1$, then $\mathrm{Sp}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$ so the subspace S_2 of $\mathbb{R}^2$ can be identified with the hyperbolic plane $(\mathcal{H}^2, ds^2)$, after rescaling the metric appropriately.

2.8.2.3. The group $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ acts by isometries on $\mathcal{H}^2 \times \mathcal{H}^2$ endowed with the produt metric and $K = \mathrm{SO}(2) \times \mathrm{SO}(2)$ fixes the point $o := (i,i) \in \mathcal{H}^2 \times \mathcal{H}62$. in this case the symmetric space G/K endowed with the G-invariant metric induced by $< X, Y >= \mathrm{tr}(XY)$ is isometric to a product of hyperbolic planes $\mathcal{H}^2 \times \mathcal{H}^2$.

2.9. Decomposition of Symmetric Spaces

2.9.1. Orthogonal Symmetric Lie Algebras. We have seen that a globally symmetric space M together with the choice of a base point $o \in M$ gives rise to a pair $(\mathfrak{g}, \Theta)$, where $\mathfrak{g}$ is the Lie algebra of (the connected component of) the group of isometries of M and Θ is the Cartan involution in Definition 2.4.11, that is the differential $\Theta = d_e\sigma$ of the involutive automorphism σ of G induced by the geodesic symmetry at o.

We denote by $\mathrm{Int}(\mathfrak{g})$ the Lie algebra of inner automorphisms of $\mathfrak{g}$. Notice that if $\mathfrak{g}$ is semisimple then all automorphisms are inner and that if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then $\mathrm{Int}(\mathfrak{g}) = \mathrm{Ad}_{\mathrm{G}}(G)$. **CHECK THAT IT IS CORRECT. p. 130?**

Definition 2.9.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$, $\mathfrak{k} \subset \mathfrak{g}$ a Lie subalgebra and K the Lie subgroup of $\mathrm{Ad}_{\mathrm{G}}(G) \leq \mathrm{GL}(\mathfrak{g})$ corresponding to the Lie subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}) \subset \mathrm{ad}_{\mathfrak{g}}(\mathfrak{g})$. The Lie algebra $\mathfrak{k}$ is called *compactly embedded in* $\mathfrak{g}$ if K is compact.

Remark 2.9.2. If G is a Lie group with Lie algebra $\mathfrak{g}$, $K < G$ is the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$, then $K = \mathrm{Ad}_{\mathrm{G}}(K)$ (as both groups are generated by $\mathrm{Ad}_{\mathrm{G}}(\exp(X))$, for $X \in \mathfrak{k}$). **TRUE ALSO IF G IS NOT LINEAR?**

Definition 2.9.3.
1. An *orthogonal symmetric Lie algebra* is a pair $(\mathfrak{l}, \varsigma)$, where $\mathfrak{l}$ is a Lie algebra over $\mathbb{R}$ and $\varsigma \in \mathrm{Aut}(\mathfrak{l})$ is an involutive automorphism of $\mathfrak{l}$ such that its set of fixed points $\mathfrak{u} := \{X \in \mathfrak{l} : \varsigma X = X\}$ is a compactly embedded subalgebra of $\mathfrak{l}$.
2. The orthogonal symmetric Lie agebra $(\mathfrak{l}, \varsigma)$ is *effective* if $\mathfrak{l} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z} \subset \mathfrak{l}$ is the center of $\mathfrak{l}$.

The prominent example of effective orthogonal symmetric Lie algebra is the pair $(\mathfrak{g}, \Theta)$ coming from a globally Riemannian symmetric space (see Teorem 2.2.7).

Definition 2.9.4. Let $(\mathfrak{l}, \varsigma)$ be an effective orthogonal symmetric Lie algebra with Killing form $B_{\mathfrak{l}}$, and let $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$ be the decomposition of $\mathfrak{l}$ into the eigenspaces of ς corresponding respectively to the $+1$ and the -1 eigenvalue.
1. If $\mathfrak{l}$ is semisimple and compact, then $(\mathfrak{l}, \varsigma)$ is called of *compact type*.
2. If $\mathfrak{l}$ is semisimple and non-compact, and moreover if $B_{\mathfrak{l}}|_{\mathfrak{u}}$ is negative definite and $B_{\mathfrak{l}}|_{\mathfrak{e}}$ is positive definite, then $(\mathfrak{l}, \varsigma)$ is called of *non-compact type*.
3. If $\mathfrak{e}$ is an Abelian ideal, then $(\mathfrak{l}, \varsigma)$ is called of *Euclidean type*.

Lemma 2.9.5. *The subspaces $\mathfrak{u}$ and $\mathfrak{e}$ are orthogonal with respect to the Killing form.*

PROOF. Let $X \in \mathfrak{u}$ and $Y \in \mathfrak{e}$ be arbitrary, so that, by definition, $\varsigma X = X$ and $\varsigma Y = -Y$. Moreover, since ς is a Lie algebra automorphism,

$$B_{\mathfrak{l}}(X, Y) = B_{\mathfrak{l}}(\varsigma X, \varsigma Y) = B_{\mathfrak{l}}(X, -Y)\,,$$

which implies that $B_{\mathfrak{l}}(X, Y) = 0$. □

Moreover, from last semester, we saw that the Killing form $B_{\mathfrak{l}}$ restricted to $\mathfrak{u}$ is negative definite, since $\mathfrak{u}$ is compactly embedded.

We say that a pair (L, U) is *associated with an orthogonal symmetric Lie algebra* $(\mathfrak{l}, \varsigma)$, if L is a connected Lie group with Lie algebra $\mathfrak{l}$, and U is a Lie subgroup of L with Lie algebra $\mathfrak{u}$. So one can define the type of a pair (L, U), according to the type of the effective orthogonal Lie algebra to which it is associated. Similarly, the type of a globally symmetric space M is defined as the type of an associated symmetric pair (G, K) naturally associated to an effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \Theta)$ as above.

Notice that, even though every choice of a base point gives rise *a priori* to a different Riemannian symmetric pair, the types of such pairs are not changed: if instead of a base point $o \in M$ we take the base point $x = g \cdot o$, for $g \in G$, then Lie algebra $\mathfrak{g}$ is the same and the involution Θ is replaced by $\mathrm{Ad}_{\mathrm{G}}(g)\Theta$.

THEOREM 2.9.6. *Let $(\mathfrak{l}, \varsigma)$ be an effective orthogonal symmetric Lie algebra. Then there exist ideals $\mathfrak{l}_0, \mathfrak{l}_+$ and $\mathfrak{l}_-$ such that*

1. *$\mathfrak{l}$ can be decomposed as a direct sum $\mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{l}_+ \oplus \mathfrak{l}_-$.*
2. *The ideals $\mathfrak{l}_0, \mathfrak{l}_+$ and $\mathfrak{l}_-$ are invariant under ς and orthogonal with respect to the Killing form $B_{\mathfrak{l}}$.*
3. *The pairs $(\mathfrak{l}_0, \varsigma|_{\mathfrak{l}_0}), (\mathfrak{l}_+, \varsigma|_{\mathfrak{l}_+})$ and $(\mathfrak{l}_-, \varsigma|_{\mathfrak{l}_-})$ are effective orthogonal symmetric Lie algebras respectively of the Euclidean type, compact type and non-compact type.*

The rest of this subsection is devoted to the sketch of the proof of this theorem.

Recall first of all from linear algebra that if E is a Euclidean space with scalar product Q and $B : E \times E \to \mathbb{R}$ is a symmetric bilinear form, then there exists an orthonormal basis $(f_1, \dots, f_n)$ (orthonormal with respect to Q) with respect to which B is diagonal. More specifically, we can write

$$B(x, y) = Q(Ax, y)\,,$$

for all $x, y \in E$, where $A \in \mathrm{End}(E)$ is symmetric. Then there exists $T \in \mathrm{O}(n, \mathbb{R})$ such that $T^{-1}AT$ is diagonal (and we write $T^{-1}AT = \mathrm{diag}(\beta_1, \dots, \beta_n)$). Thus, setting $Te_j =: f_j$, we have

$$B(f_i, f_j) = Q(Af_i, f_j) = Q(ATe_i, Te_j) = Q(T^{-1}ATe_i, e_j) = \beta_i \delta_{ij}$$

and $Q(f_i, f_j) = \delta_{ij}$ since $(f_1, \dots, f_n)$ is orthonormal.

To apply this, let us consider the adjoint subgroup $U \leq \mathrm{Ad}_{\mathfrak{l}}(\mathfrak{l})$ corresponding to $\mathrm{ad}_{\mathfrak{l}}|_{\mathfrak{u}}$. Thus U is a compact subgroup of $\mathrm{GL}(\mathfrak{l})$ that leaves $\mathfrak{e}$ invariant (since

$\mathrm{ad}_{\mathfrak{l}}(\mathfrak{u})\mathfrak{e} = [\mathfrak{u}, \mathfrak{e}] \subset \mathfrak{e}$) and hence there exists a positive definite symmetric bilinear form Q on $\mathfrak{e}$.

Let us define

$$\mathfrak{e}_0 = \sum_{\beta_j = 0} \mathbb{R} f_j\,, \qquad \mathfrak{e}_+ = \sum_{\beta_j > 0} \mathbb{R} f_j\,, \qquad \mathfrak{e}_- = \sum_{\beta_j < 0} \mathbb{R} f_j\,. \tag{2.9.1}$$

It is immediate that

LEMMA 2.9.7. (1) *$\mathfrak{e} = \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-$ is a direct sum decomposition;*

(2) *the subspaces $\mathfrak{e}_0, \mathfrak{e}_+$ and $\mathfrak{e}_-$ are orthogonal with respect to Q and $B_{\mathfrak{l}}$, and invariant under ς.*

(3) *Moreover $\mathfrak{e}_0, \mathfrak{e}_+$ and $\mathfrak{e}_-$ are invariant under U and under $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{u})$.*

PROOF. The first two assertions are obvious and the third one follows from the fact that $B_{\mathfrak{l}}$ and Q are invariant under Lie algebra automorphisms (hence under U) and hence the U-action commutes with A. Namely, for $Z \in U$,

$$\begin{aligned} Q(AX, Y) =& B_{\mathfrak{l}}(AX, Y) = B_{\mathfrak{l}}(\mathrm{Ad}_{\mathfrak{l}}(Z)AX, \mathrm{Ad}_{\mathfrak{l}}(Z)(Y)) \\ =& Q(A\mathrm{Ad}_{\mathfrak{l}}(Z)X, \mathrm{Ad}_{\mathfrak{l}}(Z)(Y)) \\ =& Q(\mathrm{Ad}_{\mathfrak{l}}(Z)^{-1}A\mathrm{Ad}_{\mathfrak{l}}(Z)X, (Y))\,, \end{aligned}$$

so that $\mathrm{Ad}_{\mathfrak{l}}(Z)^{-1}A\mathrm{Ad}_{\mathfrak{l}}(Z) = A$. Then

$$\begin{aligned} B_{\mathfrak{l}}(\mathrm{Ad}_{\mathfrak{l}}(Z)f, \mathrm{Ad}_{\mathfrak{l}}(Z)f) &= Q(A\mathrm{Ad}_{\mathfrak{l}}(Z)f, \mathrm{Ad}_{\mathfrak{l}}(Z)f) \\ &= Q(\mathrm{Ad}_{\mathfrak{l}}(Z)Af, \mathrm{Ad}_{\mathfrak{l}}(Z)f) \\ &= Q(Af, f) = B_{\mathfrak{l}}(f, f)\,, \end{aligned}$$

hence the invariance of $\mathfrak{e}_0, \mathfrak{e}_+$ and $\mathfrak{e}_-$. □

LEMMA 2.9.8. *The subspaces $\mathfrak{e}_0, \mathfrak{e}_+$ and $\mathfrak{e}_-$ satisfy the following relations:*

(1) $\mathfrak{e}_0 = \{X \in \mathfrak{l} : B_{\mathfrak{l}}(X, Y) = 0$ *for all* $Y \in \mathfrak{l}\}$.

(2) $[\mathfrak{e}_0, \mathfrak{e}] = \{0\}$ *and* $\mathfrak{e}_0$ *is an Abelian ideal in* $\mathfrak{l}$.

(3) $[\mathfrak{e}_-, \mathfrak{e}_+] = \{0\}$.

PROOF. (1) Let $\mathfrak{n} := \{X \in \mathfrak{l} : B_{\mathfrak{l}}(X, Y) = 0$ for all $Y \in \mathfrak{l}\}$. Observe that if $f \in \mathfrak{e}_0$, then $B_{\mathfrak{l}}(f, Y) = 0$ for all $Y \in \mathfrak{e}_0$ and hence, by Lemma 2.9.7, for all $Y \in \mathfrak{e}$ and, by Lemma 2.9.5, for all $Y \in \mathfrak{l}$, so that $\mathfrak{e}_0 \subset \mathfrak{n}$.

To see the reverse inclusion, observe that, since it is defined in terms of $B_{\mathfrak{l}}$, $\mathfrak{n}$ is invariant under the Lie algebra automorphism ς, so $\mathfrak{n} = (\mathfrak{n} \cap \mathfrak{u}) \oplus (\mathfrak{n} \cap \mathfrak{e})$. But since $B_{\mathfrak{l}}$ is negative definite on $\mathfrak{u}$, then $\mathfrak{n} \cap \mathfrak{u} = \{0\}$ and hence $\mathfrak{n} \subset \mathfrak{e}$. But, by definition, $\mathfrak{n} \cap \mathfrak{e}_+ = \mathfrak{n} \cap \mathfrak{e}_- = \{0\}$, so that $\mathfrak{n} \subset \mathfrak{e}_0$.

(2) Since $\mathfrak{e}_0$ is the kernel of $B_{\mathfrak{l}}$, it is an ideal. Moreover, by definition of $\mathfrak{e}$ and $\mathfrak{u}$, $[\mathfrak{e}_0, \mathfrak{e}] \subset \mathfrak{u}$. Then, by Lemma 2.9.7(3), if $X \in \mathfrak{e}_0$ and $Z \in \mathfrak{u}$, then $[X, Z] \in \mathfrak{e}_0$. It follows that if $Y \in \mathfrak{e}$, then, using (1),

$$B_{\mathfrak{l}}([X, Y], Z) = -B_{\mathfrak{l}}(Y, [X, Z]) = 0\,.$$

Since $B_{\mathfrak{l}}$ is non-degenerate on $\mathfrak{u}$ and $[X,Y]\in\mathfrak{u}$, then $[X,Y]=0$.

(3) Again, since $[\mathfrak{e}_+,\mathfrak{e}_-]\subset\mathfrak{u}$ and $B_{\mathfrak{l}}$ is non-degenerate on $\mathfrak{u}$, it suffices to show that $B_{\mathfrak{l}}(\mathfrak{u},[\mathfrak{e}_+,\mathfrak{e}_-])=0$. By the same argument as before, if $Z\in\mathfrak{u}$, $X_\pm\in\mathfrak{e}_\pm$, we have

$$B_{\mathfrak{l}}([Z,[X_-,X_+])=-B_{\mathfrak{l}}([X_-,Z],X_+)=0\,,$$

since, by Lemma 2.9.7(3), $[X_-,Z]=\in\mathfrak{e}_-$. □

We now define

$$\mathfrak{u}_+:=[\mathfrak{e}_+,\mathfrak{e}_+]\qquad \mathfrak{u}_-:=[\mathfrak{e}_-,\mathfrak{e}_-]\qquad \text{and } \mathfrak{u}_0:=\mathfrak{u}\ominus_{B_{\mathfrak{l}}}(\mathfrak{u}_+\oplus\mathfrak{u}_-)\,,$$

where the last equality denotes the orthogonal complement of $\mathfrak{u}_+\oplus\mathfrak{u}_-$ in $\mathfrak{u}$ with respect to $B_{\mathfrak{l}}$.

Lemma 2.9.9. *The subspaces $\mathfrak{u}_0,\mathfrak{u}_+,\mathfrak{u}_-$ are ideals in $\mathfrak{u}$, they are orthogonal with respect to $B_{\mathfrak{l}}$ and their direct sum is $\mathfrak{u}$.*

Proof. Again by Lemma 2.9.7(3), $[\mathfrak{u},\mathfrak{e}_\pm]\subset\mathfrak{e}_\pm$, so that, by the Jacobi indentity,

$$[\mathfrak{u}_\pm,\mathfrak{u}]=[[\mathfrak{e}_\pm,\mathfrak{e}_\pm],\mathfrak{u}]=-[[\mathfrak{e}_\pm,\mathfrak{u}],\mathfrak{e}_\pm]-[[\mathfrak{u},\mathfrak{e}_\pm],\mathfrak{e}_\pm]\subset[\mathfrak{e}_\pm,\mathfrak{e}_\pm]=\mathfrak{u}_\pm\,.$$

To see that $\mathfrak{u}_+$ and $\mathfrak{u}_-$ are orthogonal with respect to $B_{\mathfrak{l}}$, let $X_\pm,Y_\pm\in\mathfrak{e}_\pm$. Then, by $\mathrm{ad}_{\mathfrak{l}}$-invariance of $B_{\mathfrak{l}}$, we have

$$B_{\mathfrak{l}}([X_+,Y_+],[X_-,Y_-])=B_{\mathfrak{l}}(X_+,[Y_+,[X_-,Y_-])=0\,,$$

where the last equality follows from the fact that

$$[Y_+,[X_-,Y_-]=-[X_-,[Y_-,Y_+]]-[Y_-,[Y_+,X_-]]=-[X_-,0]-[Y_-,0]=0\,,$$

by Lemma 2.9.8(3). □

The following lemma will be necessary to show that some combination of the above defined subspaces are ideals.

Lemma 2.9.10. *We have:*

1. $[\mathfrak{u}_0,\mathfrak{e}_-]=[\mathfrak{u}_0,\mathfrak{e}_+]=\{0\}$.
2. $[\mathfrak{u}_-,\mathfrak{e}_0]=[\mathfrak{u}_-,\mathfrak{e}_+]=\{0\}$.
3. $[\mathfrak{u}_+,\mathfrak{e}_0]=[\mathfrak{u}_+,\mathfrak{e}_-]=\{0\}$.

Proof. (1) Let $Z\in\mathfrak{u}_0$, $X,Y\in\mathfrak{e}_\pm$. Then

$$B_{\mathfrak{l}}([Z,X],Y)=B_{\mathfrak{l}}(Z,[X,Y])=0\,,$$

since $[X,Y]\in\mathfrak{u}_\pm$ and $\mathfrak{u}_\pm$ is orthogonal to $\mathfrak{u}_0$. Since $[\mathfrak{u}_0,\mathfrak{e}_\pm]\subset\mathfrak{e}_\pm$ and $B_{\mathfrak{l}}$ restricted to $\mathfrak{e}_\pm$ is non-degenerate, then $[Z,X]=0$, that is $[\mathfrak{u}_0,\mathfrak{e}_\pm]=\{0\}$.

(2) and (3) Using the definition of $\mathfrak{u}_\pm$ and the Jacobi identity, we have $[\mathfrak{u}_\pm,\mathfrak{e}_0]=[[\mathfrak{e}_\pm,\mathfrak{e}_\pm],\mathfrak{e}_0]=[\mathfrak{e}_\pm,[\mathfrak{e}_\pm,\mathfrak{e}_0]]=\{0\}$, because of Lemma 2.9.8 (2). Likewise, $[\mathfrak{u}_\pm,\mathfrak{e}_\mp]=[[\mathfrak{e}_\pm,\mathfrak{e}_\pm],\mathfrak{e}_\mp]=[\mathfrak{e}_\pm,[\mathfrak{e}_\pm,\mathfrak{e}_\mp]]=\{0\}$, because of Lemma 2.9.8 (3). □

Now it is clear that since

$$\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e} = (\mathfrak{u}_0 \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-) \oplus (\mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-)\,,$$

to find the $\mathfrak{l}_0, \mathfrak{l}_+$ and $\mathfrak{l}_-$ we have to rearrange the direct summands.

It seems that setting

$$\mathfrak{l}_0 = \mathfrak{u}_0 \oplus \mathfrak{e}_0 \qquad \mathfrak{l}_+ = \mathfrak{u}_+ \oplus \mathfrak{e}_+ \qquad \mathfrak{l}_- = \mathfrak{u}_- \oplus \mathfrak{e}_-\,.$$

might be a good idea. In particular, it follows immediately from Lemma 2.9.10 that the $\mathfrak{l}_0, \mathfrak{l}_+$ and $\mathfrak{l}_-$ are ideals, so that, in particular, their Killing form is the restriction of the Killing form of $\mathfrak{l}$.

- The Killing form $B_{\mathfrak{l}_-} = B_{\mathfrak{l}}|_{\mathfrak{l}_-}$ is negative definite, hence $\mathfrak{l}_-$ is semisimple and compact. Thus $(\mathfrak{l}_-, \varsigma|_{\mathfrak{l}_-})$ is an effective orthogonal symmetric Lie algebra of compact type.
- The Killing form $B_{\mathfrak{l}_+} = B_{\mathfrak{l}}|_{\mathfrak{l}_+}$ is negative definite on $\mathfrak{u}_+$ and positive definite on $\mathfrak{e}_+$, hence it is non-degenerate. Thus $(\mathfrak{l}_+, \varsigma|_{\mathfrak{l}_+})$ is an effective orthogonal symmetric Lie algebra of non-compact type.
- We showed already in Lemma 2.9.8(2) that $\mathfrak{e}_0$ is an Abelian ideal. Moreover, since $\mathfrak{l}_\pm$ are semisimple, the center $\mathfrak{z}$ of $\mathfrak{l}$ must be all contained in $\mathfrak{l}_0$ and hence $\mathfrak{z}_0 = \mathfrak{z}$. Thus $\mathfrak{z}_0 \cap \mathfrak{u}_0 \subset \mathfrak{z} \cap \mathfrak{u} = \{0\}$ and hence we are left to observe that $\mathfrak{u}_0$ is compactly embedded. But this is true since $\mathfrak{u} \subset \mathfrak{l}$, $\mathfrak{u}_\pm \subset \mathfrak{l}_\pm$ are all compactly embedded and $\mathfrak{l}$ is the direct sum of the ideals $\mathfrak{l}_0 \oplus \mathfrak{l}_+ \oplus \mathfrak{l}_-$, (see [**Hel01**, Lemma V.1.6]).

Remark 2.9.11. We were a bit sloppy in the last part of the proof, in that the decomposition we proposed is valid only if $\mathfrak{e}_0 \neq \{0\}$. In fact, if $\mathfrak{e}_0 = \{0\}$, then our proposed $\mathfrak{l}_0$ would be equal to $\mathfrak{u}_0$. As a consequence, we would have that $\varsigma = Id$, which was not allowed. We hence set if $\mathfrak{e}_0 = \{0\}$:

$$\begin{array}{llll} \mathfrak{l}_0 := \{0\} & \mathfrak{l}_- := \mathfrak{u}_0 \oplus \mathfrak{u}_- \oplus \mathfrak{e}_- & \mathfrak{l}_+ := \mathfrak{u}_+ \oplus \mathfrak{e}_+ & \text{if } \mathfrak{e}_- \neq \{0\}\,; \\ \mathfrak{l}_0 := \{0\} & \mathfrak{l}_- := \{0\} & \mathfrak{l}_+ := \mathfrak{u}_- \oplus \mathfrak{u}_+ \oplus \mathfrak{e}_+ & \text{if } \mathfrak{e}_- = \{0\}\,. \end{array}$$

2.9.2. Sectional Curvature of Symmetric Spaces.

2.9.3. Decomposition.

2.9.4. Duality. We start with a couple of examples.

Example 2.9.12. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. Then $\mathfrak{sl}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. In fact, $A \in \mathfrak{sl}(n, \mathbb{C})$ if and only if $\operatorname{tr}(A) = 0$ if and only if $\Re \operatorname{tr}(A) = \Im \operatorname{tr}(A) = 0$ if and only if $\operatorname{tr} \Re(A) = \operatorname{tr} \Im(A) = 0$.

Example 2.9.13. Let $\mathfrak{g} = \mathfrak{su}(n, \mathbb{C}) = \{X \in \mathfrak{sl}(n, \mathbb{C}) :\ X^* + X = 0, \text{ where } X^* = \overline{X}^t\}$. Observe that $\mathfrak{su}(n, \mathbb{C})$ is a real Lie algebra. We claim that $\mathfrak{su}(n, \mathbb{C})^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$. In fact,

$$i\mathfrak{su}(n, \mathbb{C}) = \{iX \in \mathfrak{sl}(n, \mathbb{C}) :\ X^* + X = 0\} = \{X \in \mathfrak{sl}(n, \mathbb{C}) :\ X^* = X\}\,.$$

But for any $A \in \mathfrak{sl}(n, \mathbb{C})$ we can write

$$A = \underbrace{\frac{A - A^*}{2}}_{\mathfrak{su}(n,\mathbb{C})} + \underbrace{\frac{A + A^*}{2}}_{i\mathfrak{su}(n,\mathbb{C})},$$

so $\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n, \mathbb{C}) \oplus i\mathfrak{su}(n, \mathbb{C})$.

Example 2.9.14. Let $\mathfrak{g} = \mathfrak{o}(p, q)$. Since any two non-degenerate quadratic forms over $\mathbb{C}$ are equivalent, then $\mathfrak{o}(p, q)^{\mathbb{C}} = \mathfrak{o}(p + q, \mathbb{C})$. In particular $\mathfrak{o}(n, \mathbb{R})^{\mathbb{C}} = \mathfrak{o}(n, \mathbb{C})$ and $\mathfrak{o}(1, n - 1)^{\mathbb{C}} = \mathfrak{o}(n, \mathbb{C})$.

Definition 2.9.15. If $\mathfrak{h}$ is a complex Lie algebra, a *real form* of $\mathfrak{h}$ is a real Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}$. If $\mathfrak{h}$ is semisimple, then $\mathfrak{g}$ is called a *compact form* if $B_{\mathfrak{g}}$ is negative definite . By abuse of notation, if $\mathfrak{h}$ is a real Lie algebra, by a compact form we mean a compact form of $\mathfrak{h}^{\mathbb{C}}$.

Every semisimple Lie algebra has a compact form, [**Hel01**, Theorem III.6.3].

Let $(\mathfrak{l}, \varsigma)$ be an orthogonal symmetric Lie algebra with decomposition $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$. Let $\mathfrak{l}^*$ be the subspace of $\mathfrak{l}^{\mathbb{C}}$ defined by $\mathfrak{l}^* = \mathfrak{u} \oplus i\mathfrak{e}$. Then $\mathfrak{l}^*$ is a Lie algebra with the bracket inherited from $\mathfrak{l}^{\mathbb{C}}$

$$[X + iY, Z + iT] = [X, Z] - [Y, T] + i([X, T] + [Y.Z]),$$

and the map $\varsigma^* : \mathfrak{l}^{\mathbb{C}} \to \mathfrak{l}^{\mathbb{C}}$ defined by $\varsigma^*(X + iY) := (X - iY)$ is an involutive automorphism of $\mathfrak{l}^*$. We call $(\mathfrak{l}^*, \varsigma^*)$ the *dual* of $(\mathfrak{l}, \varsigma)$, (so that $(\mathfrak{l}, \varsigma)$ will be the dual of $(\mathfrak{l}^*, \varsigma^*)$).

Proposition 2.9.16. *Let $(\mathfrak{l}, \varsigma)$ be an orthogonal symmetric Lie algebra. Then*

(1) *The pair $(\mathfrak{l}^*, \varsigma^*)$ is an orthogonal symmetric Lie algebra.*
(2) *The pair $(\mathfrak{l}, \varsigma)$ is of non-compact type (resp. compact type) if and only if $(\mathfrak{l}^*, \varsigma^*)$ is of compact type (resp. non-compact type).*
(3) *The pair $(\mathfrak{l}_1, \varsigma_1)$ is isomorphic to $(\mathfrak{l}_2, \varsigma_2)$ if and only if $(\mathfrak{l}_1^*, \varsigma_1^*)$ is isomorphic to $(\mathfrak{l}_2^*, \varsigma_2^*)$.*

We recall that two orthogonal symmetric Lie algebra $(\mathfrak{l}_1, \varsigma_1)$ and $(\mathfrak{l}_2, \varsigma_2)$ are *isomorphic* if there exists an isomorphism $\varphi : \mathfrak{l}_1 \to \mathfrak{l}_2$ such that $\varsigma_2\varphi = \varphi\varsigma_1$.

Sketch of the proof. We will just give few comments. For the details see [**Hel01**, Proposition V.2.1]. To show (1) one has to show that if $\mathfrak{u}$ is compactly embedded in $\mathfrak{l}$, then $\mathfrak{u}$ is compactly embedded in $\mathfrak{l}^*$. Recall that this means that if $\mathrm{ad}(\mathfrak{u}) \subset \mathfrak{gl}(\mathfrak{l})$ is compactly embedded, then $\mathrm{ad}(\mathfrak{u}) \subset \mathfrak{gl}(\mathfrak{l})$ is also compactly embedded. But both $\mathfrak{gl}(\mathfrak{l})$ and $\mathfrak{gl}(\mathfrak{l})$ are real subalgebras of $\mathfrak{gl}((\mathfrak{l}^{\mathbb{C}})^{\mathbb{R}})$ and hence (1) follows.

To see (2) it is enough to observe that $B_{\mathfrak{l}}|_{\mathfrak{e}}$ is positive (resp. negative) definite if and only if $B_{\mathfrak{l}}|_{i\mathfrak{e}}$ is negative (resp. positive) definite.

(3) follows from the fact that any isomorphism between $\mathfrak{l}_1$ and $\mathfrak{l}_2$ extends to an isomorphism of complexifications. $\square$

EXAMPLE 2.9.17. Let $\mathfrak{l} = \mathfrak{so}(p+q)$ and let ς_{pq} denote the automorphism $\varsigma_{pq} : \mathfrak{gl}(p+q,\mathbb{C}) \to \mathfrak{gl}(p+q,\mathbb{C})$ defined by

$$\varsigma_{pq}(X) := I_{pq} X I_{pq}\,, \tag{2.9.2}$$

where $I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$. It is easy to check that $\varsigma_{pq}(\mathfrak{l}) = \mathfrak{l}$. Then $(\mathfrak{l}, \varsigma_{pq})$ is an orthogonal symmetric Lie algebra of compact type with decomposition $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$, where

$$\begin{aligned} \mathfrak{u} =& \left\{ X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{so}(p+q) : X_1 \in \mathfrak{so}(p), X_3 \in \mathfrak{so}(q) \right\} \\ \mathfrak{e} =& \left\{ X = \begin{pmatrix} 0 & X_2 \\ -X_2^t & 0 \end{pmatrix} \in \mathfrak{so}(p+q) : X_2 \in M_{p,q}(\mathbb{R}) \right\}. \end{aligned}$$

Hence $(\mathrm{SO(p+q)}, \mathrm{SO(p)} \times \mathrm{SO(q)})$ is a pair associated to $(\mathfrak{l}, \varsigma_{pq})$.

Now let $(\mathfrak{l}^*, \varsigma_{pq}^*)$ be the dual of $(\mathfrak{l}, \varsigma_{pq})$, that is $\mathfrak{l}^* = \mathfrak{u} \oplus i\mathfrak{e}$, with the same ς_{pq} (that is ς^* is the restriction to $\mathfrak{l}^*$ of the automorphism defined in (2.9.2). It is easy to see that the map

$$\begin{pmatrix} X_1 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix} \mapsto \begin{pmatrix} X_1 & X_2 \\ -X_2^t & X_3 \end{pmatrix} = \begin{pmatrix} -iI_p & 0 \\ 0 & I_q \end{pmatrix} \begin{pmatrix} X_1 & iX_2 \\ -iX_2^t & X_3 \end{pmatrix} \begin{pmatrix} iI_p & 0 \\ 0 & I_q \end{pmatrix}$$

is an isomorphism of $\mathfrak{l}^*$ onto $\mathfrak{so}(p,q)$. The pair associated to $(\mathfrak{l}^*, \varsigma_{pq}^*)$ is $(\mathrm{SO(p,q)}, \mathrm{SO(p)} \times \mathrm{SO(q)})$.

EXAMPLE 2.9.18. We consider now the above case but with $p=1$ and $q=3$. Let $\mathbb{H}$ be the algebra of the quaternions,

$$\mathbb{H} := \{x_0 + x_1 i + x_2 j + x_3 k : \, ij = k, \, jk = i\, ki = j, \, i^2 = j^2 = k^2 = -1\}\,.$$

The conjugation of $x = x_0 + x_2 i + x_2 j + x_3 k \in \mathbb{H}$ is $\overline{x} = x_0 - x_1 i - x_2 j - x_3 k$, and the norm is $N(x) = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, so that $x^{-1} = \overline{x}/N(x)$. The *trace* of an element $x \in \mathbb{H}$ is defined as $\mathrm{tr}(x) := x + \overline{x}$, and, by using the fact that $\overline{xy} = \overline{yx}$, it is easy to see that the trace is invariant under conjugation, namely

$$\mathrm{tr}(yxy^{-1}) = \mathrm{tr}(x)$$

for all $x, y \in \mathbb{H}$. Let $\mathbb{H}_0$ be the subspace of elements in $\mathbb{H}$ with trace 0, also called the *pure quaternions*, and G the group of elements in $\mathbb{H}$ of norm one. Observe that G is diffeomorphic to S^3. Consider the map

$$T_{x,y}(u) := xuy^{-1}\,.$$

Since $N(T_{x,y}(u)) = N(u)$, then $T_{x,y}$ preserves the norm; moreover, since G is connected, then $\{T_{x,y} : x, y \in G\}$ is connected, so that $T_{x,y} \in \mathrm{SO}(4)$. In fact, the map

$$(x,y) \mapsto T_{x,y}$$

is a homomorphism $T : G \times G \to \mathrm{SO}(4)$. Likewise, the map $\tau_x := T_{x,x}$ leaves $\mathbb{H}^0$ invariant and hence $\tau : G \to \mathrm{SO}(3)$ is a homomorphism.

It is not difficult to show that τ is surjective. In fact the tangent space to G at 1 is $T_1G = \{v : v + \overline{v} = 0\}$ and the differential $d_1\tau$ can be easily computed to obtain

$$d_1\tau(v)(x) = vx - xv\,.$$

The elements $\{i, j, k\}$ form a basis of T_1G, and $\mathbb{H}^0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$, so that from $d_1\tau(i)(i) = 0$, $d_1\tau(i)(j) = 2k$ and $d_1\tau(i)(k) = -2j$, one obtains

$$d_1\tau(i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}\,.$$

And analogous calculation for $d_1\tau(j)$ and $d_1\tau(k)$ shows that $d_1\tau$ is surjective onto $\mathfrak{so}(3)$. Observe also that $\ker\tau = \{e, -e\}$, and G is simply connected, hence G is the universal covering of SO(3).

Likewise one can show that $T : G \times G \to \mathrm{SO}(4)$ is surjective with kernel $\{(e,e), (-e,-e)\}$.

Hence we have a Lie algebra homomorphism

$$\varphi = (\tau^{-1} \times \tau^{-1}) \circ T : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(4)\,.$$

If $\varsigma : \mathfrak{so}(3) \times \mathfrak{so}(3) \to \mathfrak{so}(3) \times \mathfrak{so}(3)$ is defined by $\varsigma(X, Y) := (Y, X)$ and $\varsigma_{13} : \mathfrak{so}(4) \to \mathfrak{so}(4)$ is defined as in Example 2.9.17, then it is easy to check that $\varphi\varsigma = \varsigma_{13}\varphi$. The two orthogonal symmetric Lie algebra $(\mathfrak{so}(3) \times \mathfrak{so}(3), \varsigma)$ and $(\mathfrak{so}(4), \varsigma_{13})$ are hence isomorphic.

2.9.5. Irreducible Orthogonal Symmetric Lie Algebras.

Definition 2.9.19. Let $(\mathfrak{l}, \varsigma)$ be an orthogonal symmetric Lie algebra (with decomposition $\mathfrak{l} = \mathfrak{u} \oplus \mathfrak{e}$). We say that $(\mathfrak{l}, \varsigma)$ is *irreducible* if

(1) $\mathfrak{l}$ is semisimple and $\mathfrak{u}$ contains no non-trivial ideals of $\mathfrak{l}$, and
(2) $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{u})$ acts irreducibly on $\mathfrak{e}$.

Proposition 2.9.20. *Let $(\mathfrak{l}, \varsigma)$ be an orthogonal symmetric Lie algebra. Assume that $(\mathfrak{l}, \varsigma)$ is semisimple and $\mathfrak{u}$ has no non-trivial ideals of $\mathfrak{l}$. Then there are ideals $\mathfrak{l}_i$ of $\mathfrak{l}$ such that:*

(1) $\mathfrak{l} = \sum_j^{\oplus} \mathfrak{l}_j$;
(2) *the $\mathfrak{l}_j$ are pairwise orthogonal with respect to the Killing form $B_{\mathfrak{l}}$ and are ς-invariant;*
(3) *the $(\mathfrak{l}_j, \varsigma_j)$ are irreducible orthogonal symmetric Lie algebras.*

We will not prove this, but just indicate how the proof should go. We refer back to the proof of Theorem 2.9.6. Let

$$\mathfrak{e} = \sum_j f_j$$

be the decomposition of $\mathfrak{e}$ into irreducible subspaces of A (see (2.9.1)). Then (see Lemma 2.9.7) the f_j are orthogonal with respect to Q and $B_{\mathfrak{l}}$, and invariant under

ς, and moreover they are invariant under U and under $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{u})$. Thus one obtains a decomposition

$$\mathfrak{e} = \sum_j \mathfrak{e}_j \, ,$$

where the $\mathfrak{e}_j$ are invariant and irreducible under $\mathrm{ad}_{\mathfrak{l}}(\mathfrak{u})$. One can then set $\mathfrak{u}_j := [\mathfrak{e}_j, \mathfrak{e}_j]$ and $\mathfrak{l}_j := \mathfrak{u}_j \oplus \mathfrak{e}_j$ and argue as in the proof of Theorem 2.9.6.

THEOREM 2.9.21 ([**Hel01**, Theorem VIII.5.3 and Theorem VIII.5.4]). *The following diagram describes all irreducible orthogonal symmetric Lie algebras. There is moreover a correspondence between irreducible orthogonal symmetric Lie algebras on the left and their dual counterpart on the right.*

COMPACT TYPE	*NON-COMPACT TYPE*
$(\mathfrak{l}, \varsigma)$, *where* $\mathfrak{l}$ *is compact and simple and* ς *is any involutive automorphism*	$(\mathfrak{l}, \varsigma)$, *where* $\mathfrak{l}$ *is a non-compact simple Lie algebra over* $\mathbb{R}$, ς *is an involutive automorphism and* $\mathfrak{u}$ *is compactly embedded*
$(\mathfrak{l}, \varsigma)$, *where* $\mathfrak{l}$ *is a compact Lie algebra direct sum* $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$ *of simple ideals which are interchanged by the involutive automorphism* ς	$(\mathfrak{l}, \varsigma)$, *where* $\mathfrak{l} = \mathfrak{g}^{\mathbb{R}}$ *with* $\mathfrak{g}$ *a simple Lie algebra over* $\mathbb{C}$. *Here* ς *is the conjugation with respect to a maximal compactly embedded subalgebra.*

CHAPTER 3

Symmetric Spaces of Non-Compact Type

3.1. Introduction

From now on we will consider only symmetric spaces of non-compact type. We have shown in Theorem 2.9.6 that these have non-positive sectional curvature. We know already that the connected component of the isometry group is semisimple. More precisely, we have the following:

PROPOSITION 3.1.1. *The connected component of the identity of the isometry group of a globally symmetric space of non-compact type is a semisimple Lie group with trivial center and no compact factors.*

SKETCH OF THE PROOF. If there were a non-trivial center, then each central element would commute with the geodesic symmetries. But this would contradict the fact that each geodesic symmetry has a unique fixed point. If there were compact factors, then the curvature on these compact factors would be positive, hence the symmetric space would not be of compact type. $\square$

In this chapter M is a globally symmetric space of non-compact type, $G = \mathrm{Iso}(M)^\circ$ and $K := \mathrm{Stab}_G(o) < G$ is the compact stabilizer of a base point $o \in M$. Moreover $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition, $d_e\pi : \mathfrak{p} \to T_oM$ is the vector space isomorphims in Theorem 2.5.1, where $\pi : G \to M$ is the natural projection. The Riemannian structure on M will be the one induced by the Killing form $B_{\mathfrak{g}}|_{\mathfrak{p}}$. The fact that this assumption is not restrictive follows from the following result:

PROPOSITION 3.1.2. *If $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ and Q is any bilinear form on $\mathfrak{g}$ that is $\mathrm{ad}_{\mathfrak{g}}$-invariant, then there exists a constant c such that $Q = cB_{\mathfrak{g}}$.*

If $\mathfrak{g}$ is a real Lie algebra, the same assertion hold for any symmetric bilinear form.

PROOF. Let us first assume that $\mathfrak{g}$ is a complex Lie algebra. Since $B_{\mathfrak{g}}$ is non-degenerate, then there exists $A \in \mathrm{End}(\mathfrak{g})$ such that $Q(X,Y) = B_{\mathfrak{g}}(AX, Y)$. Since $\mathfrak{g}$ is simple, then $\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an irreducible representation (because otherwise any invariant subspace would be an ideal). So, by Schur's lemma, it suffices to show that A commutes with $\mathrm{ad}_{\mathfrak{g}}(X)$, for all $X \in \mathfrak{g}$, because then $A = cId$ and the assertion is proven.

Since $B_{\mathfrak{g}}$ is non-degenerate, to see that $(A\,\mathrm{ad}_{\mathfrak{g}}(X))Y = (\mathrm{ad}_{\mathfrak{g}}(X)A)Y$ it suffices to show that

$$B_{\mathfrak{g}}((A\,\mathrm{ad}_{\mathfrak{g}}(X))Y, Z) = B_{\mathfrak{g}}((\mathrm{ad}_{\mathfrak{g}}(X)A)Y, Z)$$

for all $Y, Z \in \mathfrak{g}$. But

$$\begin{aligned} B_{\mathfrak{g}}((A \operatorname{ad}_{\mathfrak{g}}(X))Y, Z) &= Q(\operatorname{ad}_{\mathfrak{g}}(X)Y, Z) = -Q(Y, \operatorname{ad}_{\mathfrak{g}}(X)Z) \\ &= -B_{\mathfrak{g}}(AY, \operatorname{ad}_{\mathfrak{g}}(X)Z) = B_{\mathfrak{g}}((\operatorname{ad}_{\mathfrak{g}}(X)A)Y, Z) . \end{aligned}$$

□

3.2. Properties of the Stabilizer of a Point in M

The goal of this section is to show that if M is a Riemannian symmetric space of non-compact type and $M \cong G/K$, then $K < G$ is a maximal compact subgroup.

More specifically:

PROPOSITION 3.2.1. *Let M be a Riemannian symmetric space of non-compact type, and $G = \operatorname{Iso}(M)^\circ$. Then:*

1. *If $o \in M$ is any point, then $\operatorname{Stab}_G(o)$ is a maximal compact subgroup of G.*
2. *If K is a maximal compact subgroup of G, then $K = \operatorname{Stab}_G(p)$ for some $p \in M$ and hence any two maximal compact subgroups are conjugated by an element $g \in G$.*

The proof relies upon Cartan fixed point theorem:

THEOREM 3.2.2 (Cartan fixed point theorem). *Let M be a complete simply connected manifold of non-positive curvature and let $H < \operatorname{Iso}(M)$ be a group of isometries with a bounded orbit. Then H fixes a point $p \in M$.*

We assume for the moment Cartan theorem and proceed to prove the result about maximal compact subgroups.

PROOF OF PROPOSITION 3.2.1. (1) Let $K' < \operatorname{Iso}(M)^\circ$ a compact subgroup such that $K < K'$. By Cartan fixed point theorem there exists $p \in M$ such that $K' \leq \operatorname{Stab}_G(p)$. If $g \in G$ is such that $gp = o$, then

$$gKg^{-1} < gK'g^{-1} \leq g \operatorname{Stab}_G(p) g^{-1} = \operatorname{Stab}_G(gp) = \operatorname{Stab}_G(o) = K ,$$

and hence the above are equalities, and in particular $K = K'$.

(2) If K is a maximal compact, then by Cartan fixed point theorem K fixes a point $q \in M$, so that $K \subset \operatorname{Stab}_G(q)$. But by (1), $\operatorname{Stab}_G(q)$ is a maximal compact, hence $K = \operatorname{Stab}_G(q)$.

The statement about conjugacy follows from the transitivity of G on M. □

In order to prove Cartan fixed point theorem we will first prove the following lemma, that states that there is a unique smallest closed ball with center in A that contains A.

LEMMA 3.2.3. *Let M be a complete simply connected manifold of non-positive sectional curvature. Let $A \subset M$ be a compact subset and, if $p \in A$, let us define*

$$r(p) := \sup\{d(p, q) : q \in A\} .$$

Then r takes its minimum value in a unique point in A.

We assume the lemma for the moment.

Proof of Theorem 3.2.2. Let $p \in M$ be a point such that the orbit $H \cdot p \subset M$ is bounded and let $A := \overline{H \cdot p}$. Then A is a H-invariant compact set, hence the unique point $p_0 \in A$ where the minimum of $r : A \to \mathbb{R}$ is attained is also H-invariant. $\square$

Finally, to conclude the proof we need to use the fact that we are in nonpositive curvature. The key point is the following:

Theorem 3.2.4 ([**Hel62**, Theorem 13.1]). *Let M be a complete Riemannian manifold of non-positive curvature, $p \in M$ and $\exp_p : T_pM \to M$ the exponential map. If $v \in T_pM$ and $\xi \in T_v(T_pM) \cong T_pM$, then Then*

$$\|d_p \exp_p(\xi)\| \geq \|\xi\| .$$

Moreover if $\sigma : [0,1] \to T_pM$ is a smooth curve, then $\mathrm{L}(\sigma) \leq \mathrm{L}(\exp_p \circ \sigma)$.

In particular if M is simply connected, then

$$d(\exp_p(v), \exp_p(w)) \geq \|v - w\| , \tag{3.2.1}$$

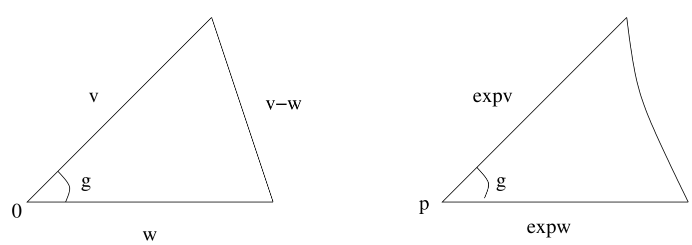
for any $v, w \in T_pM$.

Corollary 3.2.5 (Law of cosines). *If a, b, c are the length of the sides of a geodesic triangle in a non-positively curved simply connected manifold and γ is the angle opposite to the side of length c, then*

$$c^2 \geq a^2 + b^2 - 2ab\cos\gamma .$$

Proof. The law of cosines in $\mathbb{R}^n \cong T_pM$ applied to the triangle with sides $v, w \in T_pM$ such that $\sphericalangle_p(v, w) = \gamma$, $\|v\| = a$ and $\|w\| = b$, gives us that

$$\|v - w\|^2 = a^2 + b^2 - 2ab\cos\gamma .$$



Since the exponential map is an isometry along rays at $p \in M$ and since $\sphericalangle_p(v, w) = \sphericalangle_p(\exp_p(v), \exp_p(w))$, we conclude the assertion from (3.2.1) in the above theorem. $\square$

Remark 3.2.6. Theorem 3.2.4 proves that a Riemannian globally symmetric space of noncompact type is a *CAT(0)* space, that is a geodesic metric space in which triangles are "thinner" that triangles with geodesic sides of the same length in Euclidean space.

An analogous notion of *CAT(κ)* space can be given by comparison with triangles in a "model space" M_κ of curvature κ.

The CAT(κ) space generalize the notion of manifold to a purely metric setting. For more about this notion see [**BH99**].

Proof of Lemma 3.2.3. Since $|r(p_1)-r(p_2)| \leq d(p_1,p_2)$, the function $r: A \to \mathbb{R}$ is continuous and hence it attains a minimum on the compact set A.

Let us assume that the minimum r_{min} of r is attained on the two points $p_1, p_2 \in A$ and let p_0 be the midpoint of the geodesic segment joining p_1 and p_2. Let $q \in A$ be any point in A. Since $\pi = \sphericalangle_{p_0}(p_1,q) + \sphericalangle_{p_0}(p_1,q)$, where $\sphericalangle_{p_0}(p_1,q)$ denotes the angle at p_0 subtended by p_1 and q (that is the angle between the tangent vectors at p_0 of the two geodesics joining p_0 with p_1 and q) then one of the two angles $\sphericalangle_{p_0}(p_1,q)$ or $\sphericalangle_{p_0}(p_1,q)$ must be $\geq \pi/2$. By relabelling p_1 and p_2 if necessary, let us assume that $\sphericalangle_{p_0}(p_1,q) \geq \pi/2$. From the law of cosines applied to the geodesic triangle with vertices in p_0, p_1, q, we obtain that

$$\begin{aligned} d^2(q,p_1) \geq & d^2(q,p_0) + d^2(p_1,p_0) - 2d(q,p_0)d(p_1,p_0)\cos(\sphericalangle_{p_0}(p_1,q)) \\ \geq & d^2(q,p_0) + d^2(p_1,p_0) > d^2(q,p_0)\,. \end{aligned}$$

Hence

$$d(q,p_0) < \max\{d(q,p_1), d(q,p_2)\} \leq r_{min}\,,$$

which implies, by compactness of A, that

$$r(p_0) = \sup\{d(q,p_0) :\ q \in A\} < r_{min}\,.$$

This is a contradiction and hence there is unique point in A where the minimum is attained. □

3.3. Flats and Rank

Definition 3.3.1. A *k-flat* in M is a totally geodesic k-dimensional submanifold isometric to $\mathbb{R}^k$. The *rank* $\mathrm{rk}(M)$ of M is defined as

$$\mathrm{rk}(M) := \max\{k \in \mathbb{N} :\ \text{there exist a } k-\text{flat in } M\}\,.$$

If r is the rank of M, an r-flat is a *maximal* flat.

Notice that a 1-flat is nothing but a geodesic. Moreover, if the sectional curvature of the symmetric space is strictly negative, then the symmetric space must be of rank one. The rank one symmetric spaces of non-compact type are exactly the real, complex and quaternionic hyperbolic spaces and the hyperbolic plane over the Cayley numbers. All other symmetric spaces of non-compact type are of rank greater than or equal to two and hence have flats that are totally geodesic. The existence of these flats gives rise to completely different phenomena.

ly geodesic?

We want to address now the issue of trying to understand the algebraic counterpart of flats. Recall that, according to Theorem 2.7.2, a flat F is of the form $F = \exp \mathfrak{n} \cdot o$, where $\mathfrak{n} \subset \mathfrak{p}$ is a Lie triple system. Moreover the sectional curvature restricted to F is zero, so that for all $X, Y \in \mathfrak{n} \cong T_oF$ such that $B_{\mathfrak{g}}(X,Y) = 0$, $B_{\mathfrak{g}}(X,X) = B_{\mathfrak{g}}(Y,Y) = 1$, we have

$$0 = K(S) = B_{\mathfrak{g}}([X;Y],[X;Y])\,,$$

where S is the plane spanned by X, Y. Since $B_{\mathfrak{g}}$ is negative definite, it must be that $[X,Y] = 0$, so that $\mathfrak{n}$ must be abelian.

If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace of dimension $r = \mathrm{rk}(M)$, then $F := \exp \mathfrak{a} \cdot o$ is a maximal flat in M. Since G acts by isometries, on M, gF is also a flat for all $g \in G$. In fact, it is also true that every flat in M is a translate of a flat through $o \in M$.

EXAMPLE 3.3.2. We have seen that if $M = \mathrm{SL(n,\mathbb{R})/SO(n,\mathbb{R})}$, then $\mathfrak{p} = \mathrm{sym_0(n)} = \{\mathrm{X} \in \mathfrak{sl}(\mathrm{n},\mathbb{R}) :\, \mathrm{X} = \mathrm{X^t}\}$. A maximal abelian subspace of $\mathfrak{p}$ is

$$\mathfrak{a} := \left\{ \mathrm{diag}(t_1, \dots, t_n) :\, t_j \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \right\}.$$

We have seen that $\mathrm{SL(n,\mathbb{R})/SO(n,\mathbb{R})}$ is the set $\mathrm{Pos}_1(n)$ of positive matrices with determinant 1, with the action $g \cdot p = g^t p g$ for $p \in \mathrm{Pos}_1(n)$, $g \in \mathrm{SL(n,\mathbb{R})}$ and with base point $Id_n \in \mathrm{Pos}_1(n)$. Hence a maximal flat is

$$\begin{aligned} F = \exp \mathfrak{a} \cdot o &= \left\{ \mathrm{diag}(e^{2t_1}, \dots, e^{2t_n}) :\, t_j \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \right\} \\ &= \left\{ \mathrm{diag}(\lambda_1, \dots, \lambda_n) :\, \lambda_j > 0, \prod_{j=1}^n \lambda_j = 1 \right\} \end{aligned}$$

and $\mathrm{rk}(M) = n - 1$.

EXERCISE 3.3.3. Show that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.

EXAMPLE 3.3.4. Let $M = \mathrm{SO(p,q)/SO(p) \times SO(q)}$, for $p \leq q$. A maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix} :\, A = (a_{ij}) \in M_{p \times q}(\mathbb{R}),\, a_{ij} = 0 \text{ if } i \neq j \right\},$$

so $\mathrm{rk(SO(p,q)/SO(p) \times SO(q)) = min\{p,q\}}$.

EXAMPLE 3.3.5. If $M = \mathrm{Sp(2q,\mathbb{R})/(SO(2q) \cap Sp(2q,\mathbb{R}))}$, then a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} :\, A = \mathrm{diag}(t_1, \dots, t_q),\, t_j \in \mathbb{R} \right\}.$$

and hence $\mathrm{rk}(M) = q$.

Let ω be the standard symplectic form on $\mathbb{R}^{2q}$ given by $\begin{pmatrix} 0 & -Id_q \\ Id_q & 0 \end{pmatrix}$. The set S_{2q} of ω-compatible complex structures on the symplectic vector space $(\mathbb{R}^{2q}, \omega)$ is diffeomorphic to $\mathrm{Sp}(2\mathrm{q}, \mathbb{R})/(\mathrm{SO}(2\mathrm{q}) \cap \mathrm{Sp}(2\mathrm{q}, \mathbb{R}))$, with the isomorphism corresponding to the choice of base point $J_0 \in \mathrm{Sp}(2\mathrm{q}, \mathbb{R})$, where

$$J_0 := \begin{pmatrix} 0 & -Id_q \\ Id_q & 0 \end{pmatrix} .$$

Since the action of $\mathrm{Sp}(2\mathrm{q}, \mathbb{R})$ on S_{2q} is by conjugation, then

$$\begin{aligned} F &= \exp \mathfrak{a} \cdot o \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & -Id_q \\ Id_q & 0 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} : \, A = \mathrm{diag}(e^{t_1}, \dots, e^{t_n}), \, t_j \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 0 & -A^2 \\ A^{-2} & 0 \end{pmatrix} : \, A = \mathrm{diag}(e^{t_1}, \dots, e^{t_n}), \, t_j \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 0 & \mathrm{diag}(-\lambda_1, \dots, -\lambda_n) \\ \mathrm{diag}(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}) & 0 \end{pmatrix} : \, \lambda_j > 0 \right\} . \end{aligned}$$

Example 3.3.6. A maximal flat in $\mathcal{H}^2 \times \mathcal{H}^2$ is a set

$$\{(c_1(t_1), c_2(t_2)) : \, t_1, t_2 \in \mathbb{R}\} ,$$

where c_j is a geodesic in the j-th factor for $j = 1, 2$.

Lemma 3.3.7. *Every geodesic is contained in at least one maximal flat.*

Proof. Let $\gamma \in M$ be a geodesic. Then there exists $g \in G$ and $X \in \mathfrak{p}$ such that $\gamma(t) = g \exp tX \cdot o$, for $t \in \mathbb{R}$. If $\mathfrak{a}$ is a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ that contains X, then γ is contained in $g \exp \mathfrak{a} \cdot o$. □

Definition 3.3.8. Let $X \in \mathfrak{p}$ and let

$$\mathrm{Centr}_{\mathfrak{g}}(X) := \{Y \in \mathfrak{g} : \, [Y, X] = 0\}$$

be the centralizer of X in $\mathfrak{p}$. The vector X is called *regular* if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian and *singular* otherwise.

Notice that if $X \in \mathfrak{p}$ is singular, then $\dim(\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}) > \mathrm{rk}(M)$. The next lemma, that we will not prove, shows that regular elements always exist in any maximal abelian subspace.

Lemma 3.3.9 ([**Hel01**, Lemma V.6.3(i)]). *Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. Then there exists an element $X \in \mathfrak{a}$ such that* $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$.

Theorem 3.3.10. *If $\mathfrak{a}, \mathfrak{a}'$ are maximal abelian subspaces of $\mathfrak{p}$, there exists $k \in K$ such that $\mathfrak{a}' = \mathrm{Ad}_{\mathrm{G}}(k)\mathfrak{a}$.*

Corollary 3.3.11. *All maximal abelian subspaces of $\mathfrak{p}$ have the same dimension.*

Proof of Theorem 3.3.10. Let $H \in \mathfrak{a}$ and $H' \in \mathfrak{a}'$ be two regular element. Let $k_0 \in K$ be a critical point of the function $f : K \to \mathbb{R}$ defined by $f(k) := B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k)H, H')$. Then we have:

$$\begin{aligned}
0 = & \left.\frac{d}{dt}\right|_{t=0} f(k_0 \exp(tZ)) = \\
= & \left.\frac{d}{dt}\right|_{t=0} B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0 \exp(tZ))H, H') \\
= & \left.\frac{d}{dt}\right|_{t=0} B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)\mathrm{Ad}_{\mathrm{G}}(\exp(tZ))H, H') \\
= & B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0) \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}_{\mathrm{G}}(\exp(tZ)H), H') \\
= & B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)(\mathrm{ad}_{\mathfrak{g}} Z)H, H') \\
= & B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)[Z, H], H') \\
= & B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(k_0)Z, [\mathrm{Ad}_{\mathrm{G}}(k_0)H, H']) \,.
\end{aligned}$$

Since $\mathfrak{p}$ is $\mathrm{Ad}_{\mathrm{G}}(K)$-invariant, then $\mathrm{Ad}_{\mathrm{G}}(k_0)H \in \mathfrak{p}$ and hence $[\mathrm{Ad}_{\mathrm{G}}(k_0)H, H'] \in \mathfrak{k}$. Since $Z, \mathrm{Ad}_{\mathrm{G}}(k_0)Z, [\mathrm{Ad}_{\mathrm{G}}(k_0)H, H'] \in \mathfrak{k}$ and Z is arbitrary, it follows from the non-degeneracy of the Killing form that $[\mathrm{Ad}_{\mathrm{G}}(k_0)H, H'] = 0$, that is $\mathrm{Ad}_{\mathrm{G}}(k_0)H \in \mathrm{Centr}_{\mathfrak{g}}(H')$. Since H' is regular, this implies that $\mathrm{Ad}_{\mathrm{G}}(k_0)H \in \mathfrak{a}'$. But $\mathfrak{a}'$ is abelian, hence every element in $\mathfrak{a}'$ commutes with $\mathrm{Ad}_{\mathrm{G}}(k_0)H$ and hence every element of $\mathrm{Ad}_{\mathrm{G}}(k_0^{-1})\mathfrak{a}'$ commutes with H. Hence $\mathrm{Ad}_{\mathrm{G}}(k_0^{-1})\mathfrak{a}' \subset \mathfrak{a}$.

If we interchange now the roles of $\mathfrak{a}$ and $\mathfrak{a}'$, we obtain that there exists some $k \in K$ such that $\mathrm{Ad}_{\mathrm{G}}(k)\mathfrak{a}' \subset \mathfrak{a}'$. Thus

$$\mathrm{Ad}_{\mathrm{G}}(k)\mathrm{Ad}_{\mathrm{G}}(k_0^{-1})\mathfrak{a}' \subseteq \mathrm{Ad}_{\mathrm{G}}(k)\mathfrak{a} \subseteq \mathfrak{a}' \,,$$

which shows that $Adg(k)\mathrm{Ad}_{\mathrm{G}}(k_0^{-1})\mathfrak{a}' = \mathfrak{a}' = \mathrm{Ad}_{\mathrm{G}}(k)\mathfrak{a}$. □

Corollary 3.3.12. *The vector $X \in \mathfrak{p}$ is regular if and only if the geodesic $\gamma \subset M$ defined by $\gamma(t) = \exp(tX) \cdot o$, $t \in \mathbb{R}$, is contained exactly in one maximal flat.*

Proof. Let $X \in \mathfrak{p}$ be regular and let $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ be maximal abelian subspaces such that $\gamma \subset \exp \mathfrak{a} \cdot o$ and $\gamma \subset \exp \mathfrak{a}' \cdot o$. Since $X \in \mathfrak{a}'$ and $\mathfrak{a}'$ abelian then all elements in $\mathfrak{a}'$ commute with X and hence $\mathfrak{a}' \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$. But from Theorem 3.3.10 we know that $\dim \mathfrak{a} = \dim \mathfrak{a}'$, so that from $\mathfrak{a}' \subseteq \mathfrak{a}$ we deduce that $\mathfrak{a} = \mathfrak{a}'$.

Conversely, let us suppose that γ is contained in exactly one flat, $\gamma \subset \exp \mathfrak{a} \cdot o$, where $\mathfrak{a} \subset \mathfrak{p}$ is maximal abelian. Suppose that X is not regular, that is that $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is not maximal abelian and hence $\mathfrak{a} \subsetneq \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$. Let $X' \in \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ and $X' \notin \mathfrak{a}$ and choose $\mathfrak{a}' \subset \mathfrak{p}$ such that $X' \in \mathfrak{a}'$. Since $X' \notin \mathfrak{a}$, then $\mathfrak{a} \neq \mathfrak{a}'$. Moreover, $X' \in \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ implies in particular that $X' \in \mathrm{Centr}_{\mathfrak{g}}(X)$ and hence $[X, X'] = 0$. But this implies that $X \in \mathfrak{a}'$ and hence $\gamma \subset \exp \mathfrak{a}' \cdot o$, which is a contradiction. □

EXAMPLE 3.3.13 (Continuation of Example 3.3.2). $X \in \mathfrak{p}$ is regular if and only if all of its eigenvalues are distinct.

To illustrate this, let us look at the case $n = 3$ and at the vector $X = \mathrm{diag}(1, 1, -2) \in \mathrm{sym}_0(3)$. It is easy to check that

$$k(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K = \mathrm{SO}(3)$$

satisfies

$$\mathrm{Ad}_{\mathrm{G}}(k(\theta))X = X\,.$$

In particular there exists a one-parameter family of flats containing the geodesic $\gamma(t) := \exp(tX) \cdot o$, $t \in \mathbb{R}$.

EXAMPLE 3.3.14 (Continuation of Example 3.3.5). $X \in \mathfrak{p}$ is regular if and only if all of its eigenvalues are distinct and different from zero.

If $q = 2$, the vector $D = \mathrm{diag}(1, 1, -1, -1) \in \mathfrak{p}$ is singular, as every element in K of the form

$$k(\theta) = \begin{pmatrix} \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix} & 0 \\ 0 & \begin{matrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{matrix} \end{pmatrix}$$

for $\theta \in \mathbb{R}$ satisfies $\mathrm{Ad}_{\mathrm{G}}(k(\theta))X = X$.

Then the geodesic $\gamma(t) := \exp(tX) = \begin{pmatrix} 0 & -e^t Id_2 \\ e^{-t} Id_2 & 0 \end{pmatrix} \subset S_4$ belongs to the following one-parameter family of flats with parameter θ

$$\left\{ \begin{pmatrix} 0 & A_\theta(-\lambda_1, -\lambda_2) \\ A_\theta(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}) & 0 \end{pmatrix} : \lambda_1, \lambda_2 > 0 \right\} \subset S_4\,,$$

where

$$A_\theta(\mu_1, \mu_2) := \begin{pmatrix} \mu_1 \cos^2\theta + \mu_2 \sin^2\theta & (\mu_2 - \mu_1) \sin\theta\cos\theta \\ (\mu_2 - \mu_1) \sin\theta\cos\theta & \mu_1 \sin^2\theta + \mu_2 \cos^2\theta \end{pmatrix}\,,$$

for $\mu_1, \mu_2 > 0$.

Similarly, the vector $Y = \mathrm{diag}(1, 0, -1, 0)$ is invariant under $\mathrm{Ad}_{\mathrm{G}}(k(\theta))$ for any $k(\theta) \in K$ of the form

$$k(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}\,.$$

So for all $\theta \in \mathbb{R}$ the geodesic

$$\gamma(t) := \exp(tY) \cdot o = \begin{pmatrix} \begin{matrix} & 0 & \end{matrix} & \begin{matrix} -e^{2t} & 0 \\ 0 & -1 \end{matrix} \\ \begin{matrix} e^{-2t} & 0 \\ 0 & 1 \end{matrix} & 0 \end{pmatrix} \subset S_4$$

is contained in the flat

$$\left\{ \begin{pmatrix} 0 & 0 & -\lambda_1 & 0 \\ 0 & (\frac{1}{\lambda_2} - \lambda_2) \sin\theta\cos\theta & 0 & -\frac{1}{\lambda_2}\sin^2\theta - \lambda_2\cos^2\theta \\ \frac{1}{\lambda_1} & 0 & 0 & 0 \\ 0 & \lambda_2 \sin^2\theta + \frac{1}{\lambda_2}\cos^2\theta & 0 & (\lambda_2 - \frac{1}{\lambda_2})\sin\theta\cos\theta \end{pmatrix} : \lambda_1, \lambda_2 > 0 \right\}$$

Example 3.3.15 (Continuation of Example 3.3.6). We can write a geodesic in $\mathcal{H}^2 \times \mathcal{H}^2$ as

$$\gamma(t) = (c_1(t\cos\theta), c_2(t\sin\theta)) ,$$

where c_j is a geodesic in the j-th factor, $j = 1, 2$ and $\theta \in [0, \pi/2]$. Then γ is regular if $\theta \in (0, \pi/2)$ and singular if $\theta = 0, \pi/2$. So γ is singular if and only if the projection onto one factor is one point.

3.4. Roots and Root Spaces

If $\Theta : \mathfrak{g} \to \mathfrak{g}$ is the Cartan involution, we can define on $\mathfrak{g} \times \mathfrak{g}$ the following positive definite bilinear form

$$\langle\langle X, Y \rangle\rangle := -B_{\mathfrak{g}}(X, \Theta(Y)) .$$

Notice that the restriction of this form to $\mathfrak{p}$ coincides with the Killing form.

Lemma 3.4.1. *The operator* $\operatorname{ad}_{\mathfrak{g}} X$ *is self-adjoint with respect to* $\langle\langle \cdot, \cdot \rangle\rangle$, *for every* $X \in \mathfrak{p}$.

Proof. We need to show that if $X \in \mathfrak{p}$, $Y, Z \in \mathfrak{g}$, then

$$\langle\langle (\operatorname{ad}_{\mathfrak{g}} X)Y, Z \rangle\rangle = \langle\langle Y, (\operatorname{ad}_{\mathfrak{g}} X)Z \rangle\rangle .$$

This is a simple verification. In fact, since $\Theta(X) = -X$, then

$$\begin{aligned} \langle\langle (\operatorname{ad}_{\mathfrak{g}} X)Y, Z \rangle\rangle &= -B_{\mathfrak{g}}((\operatorname{ad}_{\mathfrak{g}} X)(Y), \Theta(Z)) = B_{\mathfrak{g}}(Y, (\operatorname{ad}_{\mathfrak{g}} X)\Theta(Z)) \\ &= B_{\mathfrak{g}}(Y, [X, \Theta(Z)]) = B_{\mathfrak{g}}(Y, [-\Theta(X), \Theta(Z)]) \\ &= -B_{\mathfrak{g}}(Y, \Theta((\operatorname{ad}_{\mathfrak{g}} X)Z)) = \langle\langle Y, (\operatorname{ad} X)Z \rangle\rangle . \end{aligned}$$

□

It follows from the above lemma that if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace, then $\{\operatorname{ad}_{\mathfrak{g}} X : X \in \mathfrak{a}\}$ is a commuting family of self-adjoint operators and we can consider the following

Definition 3.4.2. A linear map $\alpha : \mathfrak{a} \to \mathbb{R}$ is called a *root* of the pair $(\mathfrak{g}, \mathfrak{a})$ if

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : (\mathrm{ad}_\mathfrak{g} H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\} \neq \{0\}.$$

The subspace $\mathfrak{g}_\alpha$ is called a *root space*.

If $\alpha \equiv 0$, then $\mathfrak{a} \subseteq \mathfrak{g}_0 = \mathrm{Centr}_\mathfrak{g}(\mathfrak{a})$. If

$$\Sigma := \{\alpha : \alpha \text{ is a non-trivial root of } (\mathfrak{g}, \mathfrak{a})\}$$

denotes the (finite) set of non-trivial roots of $(\mathfrak{g}, \mathfrak{a})$, then we have a decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha .$$

The root space decomposition is a very useful tool for studying, among others, the geometry of symmetric spaces M of non-compact type. We start by characterizing regular element in terms of roots.

Lemma 3.4.3. *A vector $0 \neq H \in \mathfrak{a}$ is regular if and only if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$.*

Proof. Let us assume that $H \in \mathfrak{a} \setminus \{0\}$ is regular, that is that $\mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$ is maximal abelian. Since $\mathfrak{a}$ is maximal abelian and $\mathfrak{a} \subset \mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$, then $\mathfrak{a} = \mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$. Suppose by contradiction that there exists $\alpha \in \Sigma$ such that $\alpha(H) = 0$. Let $0 \neq X \in \mathfrak{g}_\alpha$ and let $X = X_\mathfrak{k} + X_\mathfrak{p}$ be the decomposition (in fact, $X_\mathfrak{k} = X + \Theta(X)$ and $X_\mathfrak{p} = X - \Theta(X)$). Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, we have that for all $A \in \mathfrak{a}$

$$\mathrm{ad}_\mathfrak{g}(A)(X_\mathfrak{p}) = \alpha(A)X_\mathfrak{k} . \tag{3.4.1}$$

If $\alpha(H) = 0$, then $\mathrm{ad}_\mathfrak{g}(H)(X_\mathfrak{p}) = 0$, that is $X_\mathfrak{p} \in \mathrm{Centr}_\mathfrak{g}(H)$. But since $X_\mathfrak{p} \in \mathfrak{p}$ and $\mathfrak{a} \subset \mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$, then $X_\mathfrak{p} \in \mathfrak{a}$. Hence, again from (3.4.1), for all $A \in \mathfrak{a}$,

$$0 = \mathrm{ad}_\mathfrak{g}(A)(X_\mathfrak{p}) = \alpha(A)X_\mathfrak{k} .$$

But since $\alpha \not\equiv 0$ on $\mathfrak{g}_\alpha$, then $X_\mathfrak{k} = 0$, so that $X = X_\mathfrak{p} \in \mathfrak{a}$. Thus $\mathfrak{g}_\alpha \subset \mathfrak{a}$, which is a contradiction, as $\mathfrak{a} \subset \mathfrak{g}_0$.

Conversely, let us suppose that $\mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$ is not maximal abelian, that is $\mathfrak{a} \subsetneq \mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$. Then there exists $Y \in \mathrm{Centr}_\mathfrak{g}(H) \cap \mathfrak{p}$ but $Y \notin \mathfrak{a} = \mathrm{Centr}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{p}$. Let Y_α denote the projection of Y on the root subspace $\mathfrak{g}_\alpha$, so that $Y = Y_0 \oplus \sum_{\alpha \in \Sigma} Y_\alpha$, where $Y_0 \in \mathrm{Centr}_\mathfrak{g}(\mathfrak{a}) \subset \mathrm{Centr}_\mathfrak{g}(H)$. Then

$$0 = [H, Y] = [H, \sum_{\alpha \in \Sigma} Y_\alpha] = \sum_{\alpha \in \Sigma} [H, Y_\alpha] = \sum_{\alpha \in \Sigma} \alpha(H) Y_\alpha .$$

If $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, then we would have that $Y_\alpha = 0$ for all $\alpha \in \Sigma$, that is $Y = Y_0 \in \mathrm{Centr}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{p} = \mathfrak{a}$, that is a contradiction. $\square$

We denote by $\mathfrak{a}_{\mathrm{reg}}$ the set of regular element in $\mathfrak{a}$.

Corollary 3.4.4.

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker \alpha .$$

DEFINITION 3.4.5. Let $\mathfrak{a}$ be a maximal abelian subalgebra. A connected component of $\mathfrak{a}_{\mathrm{reg}}$ is called a *Weyl chamber* in $\mathfrak{a}$.

Note that a Weyl chamber is an open cone in Euclidean space, as it is the complement of a collection of hyperplanes

$$\{X \in \mathfrak{g} : \alpha(X) = 0\}.$$

It is easy to see that a Weyl chamber can also be described as the equivalence classes in $\mathfrak{a}$ of the equivalence relation

$$H_1 \sim H_2 \qquad\qquad \alpha(H_1)\alpha(H_2) > 0, \text{ for all } \alpha \in \Sigma.$$

We denote by E_{ij} the matrix whose (i,j)-th matrix coefficient is 1 and all other are 0.

EXAMPLE 3.4.6 (Continuation of Example 3.3.2 and Example 3.3.13). Let $G/K = SL(n,\mathbb{R})/\mathrm{SO(n)}$ and $H_j := E_{jj} - E_{j+1,j+1}$. Then $(E_{ij}, i \neq j, H_1, \dots, H_{n-1})$ is a basis for $\mathfrak{g}$. If $H = \mathrm{diag}(t_1, \dots, t_n) = \sum_{j=1}^n t_j E_{jj} \in \mathfrak{a}$, then it is easy to check that

$$\mathrm{ad}_{\mathfrak{g}}(H)(E_{ij}) = [H, E_{ij}] = (t_i - t_j)E_{ij},$$

and

$$\mathrm{ad}_{\mathfrak{g}}(H)(H_j) = 0$$

for all i,j. Thus there are $n(n-1)$ non-zero roots $\{\alpha_{ij}\}_{i\neq j}$, given by

$$\alpha_{ij}(A) = A_{ii} - A_{jj},$$

and $n(n-1)$ one-dimensional root spaces $\mathfrak{g}_{ij} := \mathfrak{g}_{\alpha_{ij}}$ spanned by E_{ij} for $i \neq j$. The space $\mathfrak{a}$ is spanned by $(H_1, \dots, H_{n-1})$ and $\mathfrak{g}_0 = \mathfrak{a}$. In particular we can write

$$\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{a} \oplus \sum_{i \neq j} \mathbb{R}E_{ij}.$$

We show now that there is a one-to-one correspondence between the Weyl chambers of $\mathfrak{a}$ and the elements of the permutation group S_n in n letters. Let $A = \mathrm{diag}(t_1, \dots, t_n)$ and $B = \mathrm{diag}(\mu_1, \dots, \mu_n)$ be regular elements in $\mathfrak{a}$. Since A is regular, the λ_j are distinct and there exists a unique permutation $\sigma \in S_n$ such that

$$t_{\sigma(1)} > \cdots > t_{\sigma(n)}.$$

Similarly, since B is regular, there exists a unique permutation $\tau \in S_n$ such that

$$\mu_{\tau(1)} > \cdots > \mu_{\tau(n)}.$$

The condition that A, B determine the same Weyl chamber is exactly that they are equivalent, that is

$$(t_i - t_k)(\mu_i - \mu_k) > 0$$

for all $i \neq k$. It is not difficult to show that this holds if and only if $\sigma = \tau$, so that a Weyl chamber in $\mathfrak{a}$ is given by

$$\mathfrak{a}^+ := \{\operatorname{diag}(t_1, \dots, t_n) \in \mathfrak{a} : \sum_{i=1}^n t_i = 0,\, t_1 > t_2 > \cdots > t_n\}\,.$$

Example 3.4.7 (Continuation of the Example 3.3.4). If $G/K = \mathrm{SO}(2,3)/(\mathrm{SO}(2) \times \mathrm{SO}(3))$, then

$$\mathfrak{a} = \left\{ \begin{pmatrix} \begin{matrix} & 0 & \end{matrix} & \begin{matrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \end{matrix} \\ \begin{matrix} t_1 & 0 \\ 0 & t_2 \\ 0 & 0 \end{matrix} & 0 \end{pmatrix} : \, t_1, t_2 \in \mathbb{R} \right\} \cong \mathbb{R}^2\}\,.$$

Let $H := H(t_1, t_2) \in \mathfrak{a}$ be a matrix as above. Then

$$\alpha_1(H) := t_1 \qquad \text{and} \qquad \alpha_2(H) := t_2$$

are two roots.

Example 3.4.8 (Continuation of the Examples 3.3.5 and 3.3.14).

3.5. Root Space Decomposition

The root spaces enjoy nice symmetry properties that we will now investigate. If $\alpha \in \Sigma$, we define $H_\alpha \in \mathfrak{a}$ to be the unique element such that

$$\alpha(H) = B_{\mathfrak{g}}(H, H_\alpha)\,.$$

We call H_α a *root vector.*

Proposition 3.5.1. *Let $\alpha, \beta \in \Sigma$. Then:*

1. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
2. $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, *and, in fact, Θ is an isomorphism for each $\alpha \in \Sigma$.*
3. *If $\alpha \in \Sigma$ is not an integer multiple of another root in Σ, then the only possible multiples of α in Σ are $\pm\alpha, \pm 2\alpha$.*
4. *Θ leaves $\mathfrak{g}_0$ invariant, hence $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}$.*
5. *If $X \in \mathfrak{a}$, then, as an endomorphism of $\mathfrak{g}_\alpha$, we have*
$$\mathrm{Ad}_{\mathrm{G}}(\exp(tX)) = \exp(t\alpha(X))\,,$$
for all $\alpha \in \Sigma$.
6. *The root spaces are orthogonal with respect to the Killing form, that is $B_{\mathfrak{g}}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$, whenever $\alpha + \beta \neq 0$.*
7. *Let $\alpha, \beta \in \Sigma$. Then there exists integers $k_1, k_2 \geq 0$, such that $\beta + n\alpha \in \Sigma$, for all $-k_2 \leq n \leq k_1$. Moreover*
$$k_2 - k_1 = 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}\,.$$

In particular,

$$\beta - 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}\alpha \in \Sigma.$$

PROOF. (1) Let $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$. Then $[H, X] = \alpha(H)X$ and $[H, Y] = \beta(H)Y$, so that

$$\begin{aligned}[H,[X,Y]] &= [[H,X],Y] + [X,[H,Y]] = [\alpha(H)X, Y] + [X, \beta(H)Y] \\ &= \alpha(H)[X,Y] + \beta(H)[X,Y] = (\alpha+\beta)(H)[X,Y]\,.\end{aligned}$$

(2) Let $X \in \mathfrak{g}_\alpha$. Then, by definition, $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}$. Since $\mathfrak{a} \subset \mathfrak{p}$, then $\Theta(H) = -H$, so that

$$[H, \Theta(X)] = -[\Theta(H), \Theta(X)] = -\Theta[H, X] = -\Theta(\alpha(H)X) = -\alpha(H)\Theta(X)\,.$$

□

THEOREM 3.5.2. *Let G be a semisimple connected Lie group. Then every finite dimensional representation of G (and of $\mathfrak{g}$) is semisimple (i.e. completely reducible).*

There are at least two proofs.

LEMMA 3.5.3. *Let G be a compact Lie group. Then any representation of G on a real or complex vector space is semisimple.*

PROOF. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. Since G is compact, we can find an invariant inner product on V, so that if W is an invariant subspace, then also its orthogonal $W^\perp$ is invariant. Now apply the same argument to W and $W^\perp$. Since V is finite dimensional, then the process comes to an end. □

Recall that, if $k = \mathbb{R}$ or $\mathbb{C}$, then $\mathfrak{sl}(2, k)$ is the Lie algebra generated by

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad ,$$

with Lie bracket

$$[X^+, X^-] = H\,, \qquad [H, X^+] = 2X^+\,, \qquad [H, X^-] = -2X^-\,.$$

THEOREM 3.5.4. *Let $k = \mathbb{R}$ or $\mathbb{C}$ and let V be a finite dimensional vector space over k. Let $\mathfrak{g} \subset \mathrm{End}(V)$ a Lie algebra isomorphic to $\mathfrak{sl}(2, k)$. Assume that V is irreducible as a $\mathfrak{g}$-module. Then V has a basis of eigenvectors of H, namely*

$$V = \sum\{V_\lambda :\ \lambda = (\dim V - 1) - 2n\,, n = 0, \dots, \dim V - 1\}\,,$$

where

$$V_\lambda := \{v \in V :\ Hv = \lambda v\}\,.$$

In particular $\dim V_\lambda = 1$.

Corollary 3.5.5. *Let $k, \mathbb{R}$ or $\mathbb{C}$ be a finite dimensional vector space over k and $\mathfrak{g}$ subset $\operatorname{End}(V)$ a Lie algebra isomorphic to $\mathfrak{sl}(2,k)$. Then V has a basis of eigenvectors of H and there exist $m1, m_2 \in \mathbb{N}$, $m \geq 1$, such that the eigenvalues λ of H on V are integers $\lambda = m_i - 2n$, where $0 \leq n, \leq m_i$. In particular V is the direct sum of exactly $\dim V_0 + \dim V_1$ irreducible submodules.*

Proof of Theorem 3.5.4. □

Now we prove that if $\alpha \in \Sigma$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum \mathfrak{g}_\alpha$, then there is a copy of $\mathfrak{sl}(2,\mathbb{R})$ in $\mathfrak{g}_\alpha \oplus \mathfrak{a} \oplus \mathfrak{g}_{-\alpha}$.

Lemma 3.5.6. *Let $X^+ \in \mathfrak{g}_\alpha$ be such that*

$$\langle\langle X^+, X^+\rangle\rangle = -B_{\mathfrak{g}}(X^+, \Theta(X^+)) = \frac{-2}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}$$

and set $X^- := \Theta(X^+)$. Then the real span of $\{X^+, X^-, H\}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, where

$$H = 2\frac{H_\alpha}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}\,.$$

Proof. We had set $\alpha(Y) := B_{\mathfrak{g}}(Y, H_\alpha)$ for all $Y \in \mathfrak{a}$, so that, in particular, $\alpha(H) = 2$. Since $H_\alpha \in \mathfrak{a}$, then, by definition of root space,

$$\begin{aligned} X^+ \in \mathfrak{g}_\alpha &\Rightarrow [H, X^+] = \alpha(H)X^+ = 2X^+ \\ X^- \in \mathfrak{g}_{-\alpha} &\Rightarrow [H, X^-] = \alpha(H)X^- = -2X^-\,. \end{aligned}$$

We need to check that $[X^+, X^-] = H$. We make the following:

Claim. $[X^+, \Theta(X^+)] = -B_{\mathfrak{g}}(X^+, \Theta(X^+))H_\alpha$.

If so, then

$$\begin{aligned} [X^+, X^-] =& -[X^+, \Theta(X^+)] = -B_{\mathfrak{g}}(X^+, \Theta(X^+))H_\alpha \\ =& \frac{2}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} H_\alpha = H\,, \end{aligned}$$

and hence we are done.

We need now to prove the claim. Let $\alpha \in \Sigma$ and let $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ be any two vectors. Then for any $Y \in \mathfrak{a}$ we have

$$\begin{aligned} \langle\langle [X_\alpha, X_{-\alpha}], Y\rangle\rangle =& -B_{\mathfrak{g}}([X_\alpha, X_{-\alpha}], \Theta(Y)) = B_{\mathfrak{g}}([X_\alpha, X_{-\alpha}], Y) \\ =& B_{\mathfrak{g}}(X_{-\alpha}, [Y, X_\alpha]) = \alpha(Y)B_{\mathfrak{g}}(X_{-\alpha}, X_\alpha)\,. \end{aligned}$$

If $Z := [X_\alpha, X_{-\alpha}] - B_{\mathfrak{g}}(X_{-\alpha}, X_\alpha)H_\alpha$, then one can check that $[Y, Z] = 0$ and hence $Z \in \mathfrak{g}_0$. Moreover, from the above equation and the definition of H_α, it follows easily that $\langle\langle Z, Y\rangle\rangle = 0$ for all $Y \in \mathfrak{a}$. But according to Proposition 3.5.1(4) we have that $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}$ and the sum is orthogonal, so that $Z \in (\mathfrak{g}_0 \cap \mathfrak{k})$.

Now observe that, since $[X_\alpha, \Theta(X_\alpha)] = [\Theta^2(X_\alpha), \Theta(X_\alpha)] = \Theta([\Theta(X_\alpha), X_\alpha]) = -\Theta([X_\alpha, \Theta(X_\alpha)])$, then

$$[X_\alpha, \Theta(X_\alpha)] \in \mathfrak{p} = \{X \in \mathfrak{g} : \Theta(X) = -X\}.$$

Moreover, since $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{a} \subset \mathfrak{p}$, then

$$Z = [X_\alpha, X_{-\alpha}] - B_\mathfrak{g}(X_{-\alpha}, X_\alpha)H_\alpha = [X_\alpha, \Theta(X_\alpha)] - B_\mathfrak{g}(X_\alpha, \Theta(X_\alpha))H_\alpha \in \mathfrak{p}.$$

Since $Z \in \mathfrak{g}_0 \cap \mathfrak{k}$, and $\mathfrak{k} \cap \mathfrak{p} = \{0\}$, then $Z = 0$, that is

$$[X_\alpha, X_{-\alpha}] = B_\mathfrak{g}(X_\alpha, \Theta(X_\alpha))H_\alpha,$$

as we wanted to show. □

DEFINITION 3.5.7. Let $\alpha, \beta \in \Sigma$. An *α-string of β* is a subset of Σ of the form

$$\{\beta + n\alpha : r \le n \le s\}.$$

An α-string is *maximal* if $\beta + (r-1)\alpha \notin \Sigma$ and $\beta + (s+1)\alpha \notin \Sigma$.

LEMMA 3.5.8. *Let $\alpha, \beta \in \Sigma$ and let $\{\beta + n\alpha : r \le n \le s\}$ a maximal α-string of β. Then*

$$2\frac{B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H_\alpha, H_\alpha)} = -(r+s).$$

PROOF. Let X^+, X^-, H be as in Lemma 3.5.6, so that $\mathfrak{g}' := \mathbb{R}\text{-span}\{X^+, X^-, H\} \simeq \mathfrak{sl}(2, \mathbb{R})$. Let

$$\mathfrak{g}'' := \sum\{\mathfrak{g}_\lambda \in \Sigma : \lambda = \beta + n\alpha,\ r \le n \le s\}.$$

Consider the action of $\mathfrak{g}'$ on $\mathfrak{g}''$, via $\mathrm{ad}_\mathfrak{g}$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, because $\alpha(H) = 2$ and because of the maximality of the α-string of β, if μ is an eigenvalue of $\mathrm{ad}_\mathfrak{g}(H)$ on $\mathfrak{g}''$, then

$$\beta(H) + 2s = (\beta + s\alpha)(H) \le \mu \le (\beta + r\alpha)(H) = \beta(H) + 2r.$$

By Corollary 3.5.5, $\beta(H) + 2s = -(\beta(H) + 2r)$, that is $-(r+s) = \beta(H)$. But by definition of H and of H_β, we have that $\beta(H) = B_\mathfrak{g}(H, H_\beta) = 2\frac{B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H-\alpha, H_\alpha)}$, hence the assertion. □

END OF THE PROOF OF PROPOSITION 3.5.1. (7) We need to show that there exists integers $k_1, k_2 \ge 0$, such that $\beta + n\alpha \in \Sigma$, for all $-k_2 \le n \le k_1$. Moreover that

$$k_2 - k_1 = 2\frac{B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H_\alpha, H_\alpha)}.$$

and in particular,

$$\beta - 2\frac{B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H_\alpha, H_\alpha)}\alpha \in \Sigma.$$

Since $\beta \in \Sigma$, let us assume that

$$\{\beta + n\alpha : \, -k_2 \leq n \leq k_1\}$$

is an α-maximal string of β and that

$$\{\beta + n\alpha : \, p \leq n \leq q\}$$

is another α-maximal string of β. By the previous lemma we have that

$$-(p+q) = 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} = -(k_1 - k_2)\,. \tag{3.5.1}$$

Note that, because of maximality, we must have that either $k_1 < p$ or $q < -k_2$. In either cases, the equality $-(p+q) = -(k_1 - k_2)$ cannot hold, hence $\{\beta + n\alpha : \, -k_2 \leq n \leq k_1\}$ is the unique maximal string. Moroever, from the right hand side of (3.5.1) we obtain that, since

$$-k_2 \leq \underbrace{-k_2 + k_1}_{(k_1 - k_2)} \leq k_1\,,$$

then

$$\beta + (k_1 - k_2)\alpha = \beta - 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}\alpha \in \mathbb{Z}\,.$$

(3) We need to show that if α is not an integer multiple of another root, then the only possible multiples of α in Σ are $\pm\alpha$ and $\pm 2\alpha$. In fact, if $\beta = k\alpha$, then by definition we have that $H_\beta = kH_\alpha$.

Looking at an α-string of β, we deduce from Lemma 3.5.8 that

$$\mathbb{Z} \ni 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} = 2\frac{B_{\mathfrak{g}}(H_\alpha, kH_\alpha)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} = 2k\,.$$

On the other hand, by considering a β-string of α, we get that

$$\mathbb{Z} \ni 2\frac{B_{\mathfrak{g}}(H_\beta, H_\alpha)}{B_{\mathfrak{g}}(H_\beta, H_\beta)} = \frac{2}{k}\,.$$

We deduce that $k = \pm, \pm 2$ are the only possibilities. $\square$

3.6. Root Systems

Given the set Σ of nonzero roots and their respective root vectors, we can associate to $\alpha \in \Sigma$ the reflection S_α with respect to the hyperplane perpendicular to H_α. Thus we can write

$$S_\alpha(A) := A - 2\frac{B_{\mathfrak{g}}(H_\alpha, A)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}H_\alpha\,.$$

It is immediate to see that $S_\alpha(H_\alpha) = -H_\alpha$, while for all $H \in \mathfrak{a} \ominus H_\alpha = \{H \in \mathfrak{a} : B_{\mathfrak{g}}(H_\alpha, H) = 0\}$, we have that $S_\alpha(H) = H$.

Proposition 3.6.1. *Let $\{H_\alpha\,, S_\alpha : \alpha \in \Sigma\}$ be as above. Then:*

(1) $\mathbb{R} - \mathrm{span}\{H_\alpha\}_{\alpha\in\Sigma} = \mathfrak{a}$.
(2) *The reflections* $\{S_\alpha : \alpha \in \Sigma\}$ *leave the set of root vectors invariant.*
(3) $2\frac{B_\mathfrak{g}(H_\alpha,A)}{B_\mathfrak{g}(H_\alpha,H_\alpha)} \in \mathbb{Z}$.

The proposition shows that the above construction is only a particular case of a more general one:

Definition 3.6.2. Let V be a finite dimensional vector space over $\mathbb{R}$ and $R \subset V$ a finite set of non-zero vectors. R is called a *root system* in V and its elements are called *root* if the following conditions are verified:

(1) R generates V;
(2) for each $\alpha \in R$ there exists a reflection s_α along α that leaves R invariant (that is a linear transformation such that $s_\alpha(\alpha) = -\alpha$ and whose fixed points are a hyperplane in V).
(3) for all $\alpha, \beta \in R$, the number $m_{\alpha,\beta}$ determined by
$$s_a\beta = \beta - m_{\alpha,\beta}\alpha$$
is an integer, $m_{\alpha\beta} \in \mathbb{Z}$.

For more details see [**Hel01**, X.3.1].

Definition 3.6.3. The set $\{H_\alpha\,, S_\alpha : \alpha \in \Sigma\}$ is called the *root system* determined by the maximal abelian subalgebra $\mathfrak{a}$.

Proof of Proposition 3.6.1. (1) Let $\mathfrak{a}' := \mathrm{span}\{H_\alpha\}_{\alpha\in\Sigma}$ and let $H \in \mathfrak{a} \ominus \mathfrak{a}'$. Then $\alpha(H) = 0$ for all $\alpha \in \Sigma$, so that $\mathrm{ad}_\mathfrak{g}(H) = 0$. But this implies that H is in the center of $\mathfrak{g}$ which is trivial, since $\mathfrak{g}$ is semisimple, hence $H = 0$.

(2) Let H_β be a root vector and $\alpha \in \Sigma$. We want to show that $S_\alpha(H_\beta)$ is also a root vector, that is that
$$\gamma(H) = B_\mathfrak{g}(S_\alpha(H_\beta), H)$$
for some $\gamma \in \Sigma$. By definition of S_α we have that
$$B_\mathfrak{g}(S_\alpha(H_\beta), H) = B_\mathfrak{g}\left(H_\beta - 2\frac{B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H_\alpha, H_\alpha)}H_\alpha, H\right) = \left(\beta - 2\frac{B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H_\alpha, H_\alpha)}\alpha\right)(H)$$
By Proposition 3.5.1(7) we have that $\beta - 2\frac{B_\mathfrak{g}(H_\alpha,H_\beta)}{B_\mathfrak{g}(H_\alpha,H_\alpha)}\alpha \in \Sigma$, hence (2) is proven.

(3) has already been proven in Lemma 3.5.8. □

Here are further properties of the root system determined by $\mathfrak{a}$. In fact, the same properties hold for a general root system, where we only have to replace $B_\mathfrak{g}$ with any inner product that is left invariant by the group of linear transformation generated by the reflections (see Definition 3.6.6).

Proposition 3.6.4. (1) *If $\mu \in \Sigma$ is not a multiple of another root, then the only multiples of μ are $\pm\mu$ and $\pm 2\mu$.*
(2) *If α and β are collinear roots, then*

(a) $0 \leq 2\frac{B_{\mathfrak{g}}(H_\alpha,H_\beta)}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)}\,2\frac{B_{\mathfrak{g}}(H_\beta,H_\alpha)}{B_{\mathfrak{g}}(H_\beta,H_\beta)} \leq 3$ *and*

(b) *moroever*

$$\begin{aligned} &B_{\mathfrak{g}}(H_\alpha, H_\beta) > 0 \Rightarrow \alpha-\beta \in \Sigma \text{ and } \beta-\alpha\in\Sigma \\ &B_{\mathfrak{g}}(H_\alpha, H_\beta) < 0 \Rightarrow \alpha+\beta \in \Sigma\,. \end{aligned}$$

Proof. (1) follows from Proposition 3.5.1(7).

(2)(a) Let us set $\mu_{\alpha,\beta} = 2\frac{B_{\mathfrak{g}}(H_\alpha,H_\beta)}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)}$. Then if $\theta_{\alpha,\beta}$ is the angle between α and β,

$$\mu_{\alpha,\beta}\mu_{\beta,\alpha} = 2\frac{\|H_\beta\|}{\|H_\alpha\|}\cos\theta_{\alpha,\beta}\,2\frac{\|H_\alpha\|}{\|H_\beta\|}\cos\theta_{\beta,\alpha} = 4\cos^2\theta_{\alpha,\beta} \leq 4\,.$$

Since the $\mu_{\alpha,\beta}$ are integers, if α and β are not collinear, than $\mu_{\alpha,\beta}\mu_{\beta,\alpha} \leq 3$.

(2)(b) Let us suppose that $B_{\mathfrak{g}}(H_\alpha, H_\beta) > 0$, so that $\mu_{\alpha,\beta}$ and $\mu_{\beta,\alpha}$ must be positive. Then either $\mu_{\alpha,\beta}$ or $\mu_{\beta,\alpha}$ must be equal to 1. If $\mu_{\alpha,\beta} = 1$, then $S_\alpha(H_\beta) = H_\beta - \mu_{\alpha,\beta}H_\alpha = H_\beta - H_\alpha \in \Sigma$ since the reflections preserve the set of root vectors. Hence $H_\beta - H_\alpha = H_\gamma$ for some $\gamma \in \Sigma$ and it must be that $\gamma = \beta-\alpha$ since $(\beta-\alpha)(H) = B_{\mathfrak{g}}(H_\beta - H_\alpha, H) = B_{\mathfrak{g}}(H_\gamma, H) = \gamma(H)$. Then also $-(\beta-\alpha) \in \Sigma$. If on the other hand $\mu_{\beta,\alpha} = 1$, then one obtains with the same reasoning that $\alpha-\beta\in\Sigma$ and hence also $\beta-\alpha\in\Sigma$.

If $B_{\mathfrak{g}}(H_\alpha, H_\beta) < 0$, since $-\beta \in \Sigma$ and $-H_\beta = H_{-\beta}$, then $B_{\mathfrak{g}}(H_\alpha, H_{-\beta}) = -B_{\mathfrak{g}}(H_\alpha, H_\beta)$ and hence the assertion follows from (2)(a) with α and $-\beta$. □

Definition 3.6.5. A root system is *reduced* if for all $\alpha \in \Sigma$, the only multiple of α in Σ are $\pm\alpha$.

Many root spaces are reduced, for example the root space associated to the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

Definition 3.6.6. The *Weyl group* W of R is the group of linear transformation of V generated by the reflections

$$W := \langle S_\alpha :\, \alpha \in \Sigma\rangle\,.$$

Again, this definition of Weyl group can be given for any root system. We will see shortly another equivalent definition that makes sense in the case of a root system coming from a symmetric space and that has a more geometric meaning.

We defined earlier the Weyl chambers as the connected components of

$$\mathfrak{a}_{\mathrm{reg}} = \mathfrak{a} \smallsetminus \bigcup_{\alpha\in\Sigma} \ker\alpha\,.$$

We also saw that any regular element $H \in \mathfrak{a}$ defines uniquely a Weyl chamber $\mathcal{C}(H)$. The next result (that we will not prove) shows that the Weyl group acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$.

PROPOSITION 3.6.7 ([**Hel01, Hum72**]). *Let $H_1, H_2 \in \mathfrak{a}$ be regular elements and $\mathcal{C}(H_1), \mathcal{C}(H_2)$ be the corresponding Weyl chambers. Then there exists an element $\varphi \in W$ such that $\varphi\mathcal{C}(H_1) = \mathcal{C}(\varphi H_1) = \mathcal{C}(H_2)$. Moreover if $\varphi \in W$ is such that $\varphi\mathcal{C}(H) = \mathcal{C}(H)$, then φ is the identity.*

3.6.1. Simple Roots and Bases.

DEFINITION 3.6.8. A subset $\Delta \subset \Sigma$ is called a *basis* for Σ if

(1) The elements of Δ form a basis of α over $\mathbb{R}$,
(2) If $\beta \in \Sigma$ is any root, then $\beta = \sum_{\alpha\in\Delta} m_\alpha \cdot \alpha$, where the coefficients m_α are integers. Moreover, either $m_\alpha \geq 0$ for all $\alpha \in \Delta$ or $m_\alpha \leq 0$ for all $\alpha \in \Delta$.

DEFINITION 3.6.9. A root $\alpha > 0$ is *simple* if it cannot be written as sum of two positive roots.

LEMMA 3.6.10. *Let $\alpha \neq \beta$ two simple roots. Then $\beta - \alpha$ is not a root and $B_{\mathfrak{g}}(H_\alpha, H_\beta) \leq 0$.*

PROOF. If $\beta-\alpha$ were to be a root $\gamma \in \Sigma$, then we could write $\beta = \alpha+\gamma$ if $\gamma > 0$ or $\alpha = \beta + \gamma$ if γ is negative, thus contradicting the simplicity of α and β.

Since $\beta - \alpha$ is not a root and β is, then in Proposition 3.5.1(7) we would have that $k_2 = 0$ and $k_1 > 0$, so that

$$0 \geq -k_1 = k_2 - k_1 = 2\frac{B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)},$$

which implies that $B_{\mathfrak{g}}(H_\alpha, H_\beta) \leq 0$. □

PROPOSITION 3.6.11. *The set $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots is a basis of the root system.*

PROOF. We first show that the simple roots are linearly independent. In fact, let us suppose that

$$\gamma := \sum m_i\alpha_i = \sum n_j\alpha_j\,,$$

where all $\alpha_i \neq \alpha_j$, $m_i \geq 0$ and $n_j \geq 0$. Then $H_\gamma = \sum m_i H_{\alpha_i} = \sum n_j H_{\alpha_j}$ and hence, by Lemma 3.6.10,

$$0 \geq B_{\mathfrak{g}}(H_\gamma, H_\gamma) = \sum m_i n_j B_{\mathfrak{g}}(H_{\alpha_i}, H_{\alpha_j}) \leq 0\,.$$

Hence $\gamma = 0$, that is a contradiction.

The fact that the simple roots span α follows easily from the observation that if $\alpha \in \Sigma$ were not simple, then one could write it as $\beta + \gamma$ and continue with the decomposition until one has only the sum of simple roots. □

THEOREM 3.6.12. *Each root system has a basis Δ and any two basis are conjugate under a unique Weyl group element. If W_s is the group of isometries of $\mathfrak{a}$ generated by $\{S_\alpha : \alpha \in \Delta\}$, then W_s contains S_β for all $\beta \in \Delta$ and hence W_s is the Weyl group.*

Sketch of the proof. We only need to find a set of simple roots, that is of positive roots and then decompose them. Let H be a regular vector of $\mathfrak{a}$ and let

$$\Sigma_H^+ := \{\alpha \in \Sigma : \alpha(H) > 0\} .$$

Then by Proposition 3.6.11 Σ_H^+ is a basis of Σ.

LOOK AT IT AGAIN Suppose now that $B' = \{\alpha_1, \dots, \alpha_l\}$ is another basis. Then the element $\gamma' = \sum_{i=1}^{l} \gamma_i$, where $B_{\mathfrak{g}}(H_{\gamma_i}, H_{\alpha_j}) = \delta_{ij}$ satisfies $B_{\mathfrak{g}}(\alpha_i, \gamma') > 0$ for all i. Moreover B' is the set of simple roots in $\Sigma_{H'_\gamma}^+$.

If $\alpha \in \Sigma$, let π_α be the hyperplane orthogonal to H_α. If $\mathcal{C}(H_\gamma) = \mathcal{C}(H_{\gamma'})$, then H_γ and $H_{\gamma'}$ are on the same side of each π_α, that is $\Sigma_{H_\gamma}^+ = \Sigma_{H'_\gamma}^+$ or, equivalently, the set of simple roots are the same. The assertion now follows from the fact that the Weyl group acts simply transitively on the Weyl chambers (Proposition 3.6.7). □

3.7. Few Words on the Classification of Root Systems

Let R be a root system in V (see Definition 3.6.2). We say that R is *irreducible* if it cannot be decomposed into two disjoint nonempty orthogonal subsets. It is easy to see that any root system can be decomposed into the union of irreducible root systems.

Let $B = \{\alpha_1, \dots, \alpha_n\}$ be a basis of R. According to the proof of Proposition 3.6.4, if $\mu_{\alpha_i,\alpha_j} = 2\frac{B_{\mathfrak{g}}(\alpha_i,\alpha_j)}{B_{\mathfrak{g}}(\alpha_i,\alpha_i)}$, then $\mu_{\alpha_i,\alpha_j}\mu_{\alpha_j,\alpha_i} \in \{0, 1, 2, 3\}$. The *Coxeter graph* of R is a graph with n vertices, where the i-th vertex is joined to the j-th one by $\mu_{\alpha_i,\alpha_j}\mu_{\alpha_j,\alpha_i}$ non-intersecting lines. If R is irreducible, then there is a unique inner product on $\langle\langle\, ,\, \rangle\rangle$ on V up to a constant factors. We can hence give weight to the j-th vertex equal to $\langle\langle \alpha_j, \alpha_j \rangle\rangle$. The diagrams so obtained are called *Dynkin diagrams* and can be completely classified (see for example [**Hel01**, X]). There are four "classical" (family of) diagrams $\mathfrak{a}_l, \mathfrak{b}_l, \mathfrak{c}_l$ and $\mathfrak{d}_l$ and five "exceptional" ones, $\mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4$ and $\mathfrak{g}_2$.

If R is not reduced, one can consider the set of *indivisible* roots, that is roots such that if $\alpha \in R$, then $\frac{1}{2}\alpha \notin R$. The set of indivisible roots form a reduced root system that can be also classified.

So far we saw that associated to any simple Lie algebra (over $\mathbb{C}$) there is an irreducible root system that determines the Lie algebra up to isomorphism. We also got a glimpse above on how one can proceed to classify the irreducible reduced root systems. Finally, given any such irreducible root system, the last step is to construct a Lie algebra corresponding to any of the above root systems.

3.8. The Weyl group from the Geometric Point of View

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition with $\mathfrak{k}$ the Lie algebra of the stabilizer K of a point $o \in M$.

Proposition 3.8.1. *Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $\{H_\alpha, S_\alpha : \alpha \in \Sigma\}$ be the corresponding root system. Fix $\alpha \in \Sigma$ and let $X \in \mathfrak{g}_\alpha$ be a vector with the* *property that* $\langle\langle X, X\rangle\rangle = 2$. *Let* $X = K_\alpha + P_\alpha$ *the decomposition of* X, *where* $K_\alpha \in \mathfrak{k}$ *and* $P_\alpha \in \mathfrak{p}$. *Then:*

(1) $[K_\alpha, P_\alpha] = H_\alpha$.
(2) *There exists* $t \in \mathbb{R}$ *such that* $\mathrm{Ad}_{\mathrm{G}}(\exp(tX_\alpha)) = S_\alpha$, *that is:*
  (a) $\mathrm{Ad}_{\mathrm{G}}(g_\alpha)$ *leaves* $\mathfrak{a}$ *invariant;*
  (b) $\mathrm{Ad}_{\mathrm{G}}(g_\alpha)(H_\alpha) = -H_\alpha$;
  (c) $\mathrm{Ad}_{\mathrm{G}}(g_\alpha)(H) = H$ *if* $B_{\mathfrak{g}}(H_\alpha, H) = 0$.

We start the proof with the following:

LEMMA 3.8.2. *With the hypotheses of Proposition 3.8.1, we have that*

(1) $[H, P_\alpha] = \alpha(H)K_\alpha$ *for all* $H \in \mathfrak{a}$;
(2) $[H, K_\alpha] = \alpha(H)P_\alpha$ *for all* $H \in \mathfrak{a}$;
(3) $B_{\mathfrak{g}}(K_\alpha, K_\alpha) = -B_{\mathfrak{g}}(P_\alpha, P_\alpha) = -1$;
(4) $[H, [K_\alpha, P_\alpha]] = 0$ *for all* $H \in \mathfrak{a}$;
(5) $B_{\mathfrak{g}}(H, [K_\alpha, P_\alpha]) = \alpha(H)$ *for all* $H \in \mathfrak{a}$.

PROOF. (1) and (2) have been already proven and used before. They follow from the fact that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, the uniqueness of the decomposition and the fact that $X \in \mathfrak{g}_\alpha$ is an eigenvector of α.

(3) If $H \in \mathfrak{a}$ is such that $\alpha(H) \neq 0$, then, using (1) and (2),

$$\alpha(H)B_{\mathfrak{g}}(K_\alpha, K_\alpha) = B_{\mathfrak{g}}([H, P_\alpha], K_\alpha) = -B_{\mathfrak{g}}(P_\alpha, \mathrm{ad}_{\mathfrak{g}}(H)(K_\alpha)) = -\alpha(H)B_{\mathfrak{g}}(P_\alpha, P_\alpha).$$

To see the explicit value of $B_{\mathfrak{g}}(K_\alpha, K_\alpha) = -B_{\mathfrak{g}}(P_\alpha, P_\alpha)$, observe that by definition of $X \in \mathfrak{g}_\alpha$ and of $\langle\langle\, , \,\rangle\rangle$, and the fact that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to the Killing form, we have that

$$\begin{aligned} 2 =& \langle\langle X, X\rangle\rangle = \langle\langle K_\alpha + P_\alpha, K_\alpha + P_\alpha\rangle\rangle = B_{\mathfrak{g}}(K_\alpha + P_\alpha, \theta(K_\alpha + P_\alpha)) \\ =& B_{\mathfrak{g}}(K_\alpha + P_\alpha, K_\alpha - P_\alpha) = B_{\mathfrak{g}}(K_\alpha, K_\alpha) - B_{\mathfrak{g}}(P_\alpha, P_\alpha). \end{aligned}$$

(4) From the Jacobi indentity

$$[H, [K_\alpha, P_\alpha]] = [[H, K_\alpha], P_\alpha] + [K_\alpha, [H, P_\alpha]] = \alpha(H)[P_\alpha, P_\alpha] + \alpha(H)[K_\alpha, K_\alpha] = 0.$$

(5) Using (3), we have

$$\begin{aligned} B_{\mathfrak{g}}(H, [K_\alpha, P_\alpha]) =& - B_{\mathfrak{g}}(H, \mathrm{ad}_{\mathfrak{g}}(P_\alpha)(K_\alpha)) = B_{\mathfrak{g}}(\mathrm{ad}_{\mathfrak{g}}(P_\alpha)(H), K_\alpha) \\ =& - \alpha(H)B_{\mathfrak{g}}(K_\alpha, K_\alpha) = \alpha(H). \end{aligned}$$

□

PROOF OF PROPOSITION 3.8.1. (1) From Lemma **??**(4) we deduce immediately that $[K_\alpha, P_\alpha] \in \mathfrak{a}$, so it is just a matter to determine exactly which element of $\mathfrak{a}$. From Lemma **??**(4) and the definition of H_α we have

$$B_{\mathfrak{g}}(H, [K_\alpha, P_\alpha]) = \alpha(H) = B_{\mathfrak{g}}(H, H_\alpha),$$

so that $[K_\alpha, P_\alpha] = H_\alpha$.

(2) From (1) and Lemma **??**(2) with $H = H_\alpha$, we have that

$$[K_\alpha, H] = -\alpha(H)P_\alpha$$

and

$$(\mathrm{ad}_{\mathfrak{g}}(K_\alpha))^2(H) = -\alpha(H)[K_\alpha, P_\alpha] = -\alpha(H)H_\alpha\,.$$

Iterating this procedure (and multiplying by powers of t), we obtain

$$(t\,\mathrm{ad}_{\mathfrak{g}}(K_\alpha))^{2n}(H_\alpha) = (-1)^n t^{2n}\alpha(H_\alpha)^n H_\alpha$$

and

$$(t\,\mathrm{ad}_{\mathfrak{g}}(K_\alpha))^{2n+1}(H_\alpha) = (-1)^n t^{2n+1}\alpha(H_\alpha)^{n+1} P_\alpha\,.$$

Hence

$$\begin{aligned}\mathrm{Ad}_{\mathrm{G}}(\exp(tK_\alpha))(H_\alpha) &= \exp(t\,\mathrm{ad}_{\mathfrak{g}}(K_\alpha))(H_\alpha) = \sum_{n=0}^{\infty}\frac{(t\,\mathrm{ad}_{\mathfrak{g}}(K_\alpha))^n}{n!}(H_\alpha)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}\alpha(H_\alpha)^n}{(2n)!}(H_\alpha) + \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}\alpha(H_\alpha)^{n+1}}{(2n+1)!}(P_\alpha)\\ &= \cos(t\sqrt{\alpha(H_\alpha)})H_\alpha + \sqrt{\alpha(H_\alpha)}\sin(t\sqrt{\alpha(H_\alpha)})P_\alpha\,.\end{aligned}$$

This means that $\mathrm{Ad}_{\mathrm{G}}(\exp(tK_\alpha))$ rotates the vector H_α in the plane spanned by H_α and P_α. In particular if $t_0\sqrt{\alpha(H_\alpha)} = \pi$, then $\mathrm{Ad}_{\mathrm{G}}(\exp(t_0K_\alpha))(H_\alpha) = -H_\alpha$.

Let now $H \in \mathfrak{a}$ such that $0 = B_{\mathfrak{g}}(H, H_\alpha) = \alpha(H)$. It follows from Lemma **??**(2) that $[H, K_\alpha] = 0$, and hence

$$\mathrm{Ad}_{\mathrm{G}}(\exp(tK_\alpha))(H) = \exp(t\,\mathrm{ad}_{\mathfrak{g}}(K_\alpha))(H) = \sum_{n=0}^{\infty}\frac{(t\,\mathrm{ad}_{\mathfrak{g}}(K_\alpha))^n}{n!}(H) = H\,.$$

for all $t \in \mathbb{R}$ and in particular for t_0. □

Let $A := \exp\mathfrak{a} < G$.

Proposition 3.8.3. *The following are equivalent definitions of the Weyl group:*

1. *Let $M' := \{k \in K : \mathrm{Ad}_{\mathrm{G}}(k)\mathfrak{a} = \mathfrak{a}\}$ and $M := \{k \in K : \mathrm{Ad}_{\mathrm{G}}(k)(A) = A,$ for all $A \in \mathfrak{a}\}$. Then M is normal in M' and $W = M'/M$.*
2. *$W = \mathrm{Norm}_G(A)/\mathrm{Centr}_G(A)$, where $\mathrm{Norm}_G(A) := \{g \in G : gAg^{-1} = A\}$ and $\mathrm{Centr}_G(A) := \{g \in G : gA = Ag\}$.*
3. *$W = \mathrm{Norm}_G(A) \cap K/(\mathrm{Centr}_G(A) \cap K)$, where $\mathrm{Norm}_G(A)$ and $\mathrm{Centr}_G(A)$ are as in (2).*

Sketch of the proof. We are just going to show that:

1. The groups M' and M are compact and have the same Lie algebra $\mathfrak{k}_0 := \mathfrak{k} \cap \mathfrak{g}_0 = \{X \in \mathfrak{k} : [X, H] = 0$ for all $H \in \mathfrak{a}\}$
2. If $g \in M'$, then $\mathrm{Ad}_{\mathrm{G}}(g)$ permutes the root vectors H_α and the root spaces $\mathfrak{g}_\alpha$.

It will follow that W is isomorphic to a subgroup of M'/M. The fact that it is in fact equal follows from the fact that M'/M acts simply transitively on the set of Weyl chambers, that is that if $g \in K$ is such that $\mathrm{Ad}_{\mathrm{G}}(g)\mathcal{C}(H_0) = \mathcal{C}(H_0)$ for some regular element $H_0 \in \mathfrak{a}$, then $\mathrm{Ad}_{\mathrm{G}}(g)(H) = H$ for every $H \in \mathfrak{a}$.

Let $\mathfrak{m}, \mathfrak{m}'$ be the Lie algebras of M, M' respectively. We will show that

$$\mathfrak{g}_0 \cap \mathfrak{k} \subseteq \mathfrak{m} \subseteq \mathfrak{m}' \subseteq \mathfrak{g}_0 \cap \mathfrak{k}\,,$$

so that M'/M will be discrete. Observe that M, M' are compact as they are closed subgroups of K.

Let $X \in \mathfrak{m}'$. Then $\exp(tX) \in M'$ and $\mathrm{Ad}_{\mathrm{G}}(\exp(tX))(\mathfrak{a}) \subset \mathfrak{a}$. Likewise, $\mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{a}) \subseteq \mathfrak{a}$. Let $X = X_0 \oplus \sum_{\alpha\in\Sigma} X_\alpha$ be the decomposition of X corresponding to $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$. Let $H \in \mathfrak{a}$ be a regular element. Then

$$\mathfrak{a} \ni \mathrm{ad}_{\mathfrak{g}}(X)(H) = -\sum_{\alpha\in\Sigma} \alpha(H) X_\alpha \in \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha\,;$$

since $\mathfrak{a} \cap \sum \mathfrak{g}_\alpha = \{0\}$ and $H \in \mathfrak{a}$ is regular – so that $\alpha(H) \neq 0$ – then $X_\alpha = 0$ for all $\alpha \in \Sigma$, and hence $X = X_0 \in \mathfrak{g}_0$.

Now let $X \in \mathfrak{g}_0 \cap \mathfrak{k}$ and $H \in \mathfrak{a}$ any nonzero element. Then, since $\mathrm{ad}_{\mathfrak{g}}(X)(H) = 0$, it follows that for all $t \in \mathbb{R}$,

$$\mathrm{Ad}_{\mathrm{G}}(\exp(tX))(H) = \exp(t\,\mathrm{ad}_{\mathfrak{g}}\, X)(H) = H\,.$$

Hence $\exp(tX) \in M$ and thus $X \in \mathfrak{m}$.

To see (2), if $g \in M'$, and $\alpha \in \Sigma$, let us define a linear form on $\mathfrak{a}$ by $\beta := \alpha \circ \mathrm{Ad}_{\mathrm{G}}(g)^{-1}$. It is easy to verify that

$$\mathrm{ad}_{\mathfrak{g}}(H)(\mathrm{Ad}_{\mathrm{G}}(g)(X)) = \beta(H)X\,,$$

so that $\beta \in \Sigma$ and $\mathrm{Ad}_{\mathrm{G}}(g)(\mathfrak{g}_\alpha) \subset \mathfrak{g}_\beta$. Applying the same argument to g^{-1}, we obtain that $\mathrm{Ad}_{\mathrm{G}}(g)^{-1}(\mathfrak{g}_\beta) \subseteq \mathfrak{g}_\gamma$ for some $\gamma \in \Sigma$. It follows that $\gamma = \alpha$ and $\mathrm{Ad}_{\mathrm{G}}(g)\mathfrak{g}_\alpha = \mathfrak{g}_\beta$, namely $\mathrm{Ad}_{\mathrm{G}}(g)$ permutes the root spaces.

To see that $\mathrm{Ad}_{\mathrm{G}}(g)$ also permutes the root vectors, observe that, since $\mathrm{Ad}_{\mathrm{G}}(g)$ leaves invariant the Killing form, then Hence for all $H \in \mathfrak{a}$ we have that

$$B_{\mathfrak{g}}(\mathrm{Ad}_{\mathrm{G}}(g)H_\alpha, H) = B_{\mathfrak{g}}(H_\alpha, \mathrm{Ad}_{\mathrm{G}}(g)^{-1}H) = \alpha(\mathrm{Ad}_{\mathrm{G}}(g)^{-1}H) = \beta(H) = B_{\mathfrak{g}}(H_\beta, H)\,,$$

that if $\mathrm{Ad}_{\mathrm{G}}(g)H_\alpha = H_\beta$. $\square$

CHAPTER 4

The Geometry at Infinity of a Symmetric Space of Non-Compact Type

We will give some hints to the rich geometry at infinity of a Riemannian globally symmetric space M of non-compact type. We have seen that M is a Hadamard manifold, that is a complete simply connected Riemannian manifold of non-positive sectional curvature. Therefore M is homeomorphic to $\mathbb{R}^{\dim(M)}$ and can be compactified by attaching its *geometric boundary*, which can be described, in the case of a symmetric space, much more precisely than in the case of a general Hadamard manifold. In addition to the geometric boundary, there is also the so-called *Furstenberg boundary*. The two coincide in the case of a symmetric space of rank one, but in general the Furstenberg boundary is the quotient of a dense subset of the geometric boundary.

Any two points in a symmetric space can be joined by a geodesic, but the same is not true for any two points of the boundary, unless the symmetric space is, again, of rank one.

We will see these concepts illustrated in the specific example of $\mathrm{SL}(\mathrm{n},\mathbb{R})/\mathrm{SO}(\mathrm{n})$.

4.1. Basic Definitions

Definition 4.1.1. We say that two geodesic rays γ_1, γ_2 are *equivalent* if

$$\lim_{t\to\infty} d(\gamma_1(t), \gamma_2(t)) < \infty$$

The *geometric* (or *visual*) *boundary* ∂M of M is the set of geodesic rays in M modulo this equivalence relation. If γ is a geodesic ray in M, we denote by $\gamma(\infty)$ the point in ∂M that it defines.

We give $\overline{M} := M \cup \partial M$ the *cone topology* generated by the open sets in M and the *cones* defined as follows: if $\epsilon > 0$, $R >> 1$, $p \in M$ and $\xi \in \overline{M}$, the *cone* $C^{R,\epsilon}_{p,\xi}$ is defined by

$$C^{R,\epsilon}_{p,\xi} := \{y \in \overline{M} : \, d(p,y) > R, d(c_{x,\xi}(R), c_{p,y}(R)) < \epsilon\} ,$$

where $c_{x,\xi}$ is the unique unit speed geodesic from p in the class of ξ.

The isometry group acts on ∂M by homeomorphisms: if $g \in G$ and $\xi \in \partial M$ is represented by γ, then $g \cdot \gamma$ is the class of the geodesic ray $g \cdot \gamma$ in M. Notice that this assignment is well defined, that is it does not depend on the choice of γ in the equivalence class defining ξ. In fact, if γ' is another geodesic ray with

$\gamma'(\infty) = \xi$, then $d(\gamma(t), \gamma'(t))$ is bounded as $t \to \infty$. Since g is an isometry, then $d(g \cdot \gamma(t), g \cdot \gamma'(t))$ is also bounded, hence $g \cdot \gamma$ is equivalent to $g \cdot \gamma'$.

4.2. SL(n, ℝ)/SO(n)

Given a point $\xi \in \partial M$, let $X \in \mathfrak{p}$ be the unit vector such that $\xi = \gamma(\infty)$, where $\gamma_X(t) = \exp(tX) \cdot o$. Let $\{\lambda_1(\xi), \dots, \lambda_k(\xi)\}$ be the eigenvalues of X, numbered so that $\lambda_1(\xi) > \cdots > \lambda_k(\xi)$ and let $E_j(\xi)$ be the eigenspace of X in $\mathbb{R}^n$ corresponding to the eigenvalue $\lambda_j(\xi)$ and let $V_i(\xi) := \sum_{j=1}^{i} E_j(\xi)$. We call

$$\{0\} \subset V_1(\xi) \subset V_2(\xi) \subset \cdots \subset V_k(\xi) = \mathbb{R}^n$$

a *flag*. It can easily be shown that the vector $\lambda(\xi) := (\lambda_1(\xi), \dots, \lambda_k(\xi))$ and the flag $F(\xi) := (V_1(\xi), V_2(\xi), \dots, V_k(\xi)$ satisfy the following properties:

(1) $\lambda_1(\xi) > \cdots > \lambda_k(\xi)$;
(2) $\sum_{i=1}^{k} m_i \lambda_i(\xi) = 0$, where $m_i := \dim V_i(\xi) - \dim V_{i-1}(\xi)$ (i.e. $\operatorname{tr}(X) = 0$);
(3) $\sum_{i=1}^{k} m_i \lambda_i^2(\xi) = 1$ (i.e. $\|X\| = 1$).

Conversely, if we have a vector λ and a flag F that satisfy (1), (2) and (3) or some $k \in \mathbb{N}$, $k \leq n$, then there exists a unique $\xi \in \partial M$ such that $\lambda = \lambda(\xi)$ and $F = F(\xi)$. We call the pair (λ, F) an *eigenvalue-flag* pair.

Example 4.2.1. For example if $k = n$ (that is $X = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with $\lambda_1 > \cdots > \lambda_n$, then F is a *full* or *maximal* or *regular* flag and $V_j = \operatorname{span}\{e_1, \dots, e_j\}$.

The set of eigenvalue-flag pairs (satisfying the above properties) is a model of the geometric boundary of SL(n, ℝ)/SO(n).

Now we look at the action of G on the set of eigenvalue-flag pairs. If $F = (V_1, \dots, V_k)$ is a flag and $g \in \mathrm{SL}(n, \mathbb{R})$, the assignment $gF := (gV_1, \dots, gV_k)$ defines an action of SL(n, ℝ) on the space of flags in $\mathbb{R}^n$.

Proposition 4.2.2. *Let $g \in \mathrm{SL}(n, \mathbb{R})$ and $\xi \in \partial M$ with associated eigenvalue-flag pair $(\lambda(\xi), F(\xi)$. Then the eigenvalue-flag pair associated to $g\xi$ is $(\lambda(\xi), gF(\xi))$, that is:*

(1) $\lambda(g\xi) = \lambda(\xi)$;
(2) $g \cdot F(\xi)) = F(g\xi)$.

It follows immediately from the proposition that $g\xi = \xi$ if and only if $gF(\xi) = F(\xi)$.

Example 4.2.3. If F is a full flag, then the stabilizer of the corresponding point in ∂M is the group of upper triangular matrices $U < \mathrm{SL}(n, \mathbb{R})$.

The group SL(n, ℝ) acts transitively on the space of full flags, that can hence be identified with the homogeneous space SL(n, ℝ)/U.

Definition 4.2.4. (1) Two flags $F = (V_1, \dots, V_k)$ and $F' = (V_1^*, \dots, V_r^*)$ are *in opposition* if $k = r$ and $V_j \oplus V_{r-j+1}^* = \mathbb{R}^n$ for all $1 \leq j \leq k$.

(2) The *inverse* of a flag $F = (V_1, \dots, V_k)$ is defined as $F^{-1} := (V_1^*, \dots, V_k^*)$, where V_j^* is defined as the orthogonal complement of V_{k-j+1} in $\mathbb{R}^n$.

It follows from the definition that a flag is always in opposition to its inverse.

Let $\xi \in \partial M$, $\xi = \gamma_X(\infty)$ and let $\eta = \gamma_X(-\infty)$. Since $\gamma_X(-\infty) = \gamma_{-X}(\infty)$, then it is easy to see that $F(\eta) = F^{-1}(\xi)$. Moreover it is also easy to see that F and F' are in opposition if and only if there exists $g \in \mathrm{SL}(\mathrm{n}, \mathbb{R})$ such that $gF = F$ and $gF^{-1} = F'$.

It is clear from the definition that if $F = (V_1, \dots, V_k)$ is a full flag, there exists a dense open subset $\mathcal{O}$ in the space of full flags such that F is in opposition to F' for every $F' \in \mathcal{O}$.

The following criterion in terms of eigenvalue-flag pairs will identify two points in ∂M that can be joined by a geodesic.

Proposition 4.2.5. *Let $\xi, \eta \in \partial M$ be two distinct points with corresponding eigenvalue-flag pairs $(\lambda(\xi), F(\xi))$ and $(\lambda(\eta), F(\eta))$. Then there is a geodesic γ such that $\gamma(\infty) = \xi$ and $\gamma(-\infty) = \eta$ if and only if:*

(1) *$F(\xi)$ and $F(\eta)$ are in opposition, and*
(2) *$\lambda_i(\xi) = -\lambda_{k-i+1}(\eta)$.*

Proof. We show only that the two conditions are sufficient. Let $\xi, \eta \in \partial M$. Let $\xi = \gamma_X(\infty)$ and $\zeta := \gamma_{-X}(\infty)$. Then by the discussion above there exists $g \in \mathrm{SL}(\mathrm{n}, \mathbb{R})$ such that $gF(\xi) = F(\xi)$ and $gF^{-1}(\xi) = gF(\zeta) = F(\eta)$. Since $gF(\xi) = F(\xi)$, it follows that $g\xi = \xi$, so that $\xi = g\xi$ can be joined to $g\zeta$ by the geodesic $g\gamma_X$. But by (2) we know that

$$F(\eta) = gF(\zeta) = F(g\zeta) \qquad \text{and} \qquad \lambda(\eta) = \lambda(\zeta) = \lambda(g\zeta)\,,$$

hence $\eta = g\zeta$ since they have the same eigenvalue-flag pair. □

APPENDIX A

Preliminaries

A.1. Topological Preliminaries

Definition A.1.1. Let X by a a topological space and (Y, d) a metric space. A family $\mathcal{F} \subset C(X, Y)$ of functions if *equicontinuous* is for every $x \in$X and every $\epsilon > 0$, there exists an open $U_x \subset X$ such that for all $f \in \mathcal{F}$ and all $x' \in U_x$, $d(f(x), f(x')) < \epsilon$.

Theorem A.1.2 (Ascoli–Arzelà). *Let X be a topological space and (Y, d) a metric space. Give $C(X, Y)$ the compact-open topology and let $\mathcal{F} \subset C(X, Y)$. Then:*

(1) *$\mathcal{F}$ is equicontinuous and the set $\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$ has compact closure for all $a \in X$ then $\mathcal{F}$ is relatively compact.*
(2) *The converse holds is X is locally compact and Hausdorff.*

Definition A.1.3. A *(topological) fiber bundle* $\mathcal{B}$ consists of

(1) a topological space B (*bundle space*),
(2) a topological space X (*base space*),
(3) a continuous map $P : B \twoheadrightarrow X$ (*projection*),
(4) a topological space Y (*fiber*) such that for every $x \in X$ the *fiber* $p^{-1}(x)$ must be homeomorphic to Y.

Moreover a fiber bundle is *locally trivial*, that is

(5) for every $x \in X$, there exists a neighborhood V of x and a homeomorphism
$$\varphi : V \times Y \times p^{-1}(V)$$
such that the diagram
$$\begin{tikzcd}
p^{-1}(V) \arrow[d, "p"'] & V \times Y \arrow[l, "\varphi"'] \arrow[dl, "\mathrm{pr}_1"] \\
V &
\end{tikzcd}$$
commutes, that is $p\varphi(x', y) = x'$ for all $x' \in V$ and $y \in Y$.

Finally a *cross-section* of $\mathcal{B}$ is a a continuous map $\sigma : X \to B$ that is a right inverse to p, that is such that $p \circ \sigma(x) = x$ for all $x \in X$.

We consider now the case in which B is a group, so that X ia a G-homogeneous space $X = G/H$ and $Y = H = \mathrm{Stab}_G(p)$. This is what is called a *principal* *bundle*. In this case having a *local* cross-section implies (5). In fact we can define $\varphi : V \times H \to p^{-1}(V)$ as

$$\varphi(x,h) := \sigma(x)h$$

and it is easy to verify that $p \circ \varphi(x,h) = x$. For the inverse φ^{-1} we have the formula

$$\varphi^{-1}(p^{-1}(x)) = (x, x^{-1}\sigma(p(x)))\,,$$

where we need to verify that $x^{-1}\sigma(p(x)) \in H$. In fact, by definition of σ,

$$p(x^{-1}\sigma(p(x))) = p(x^{-1})p(\sigma(p(x))) = p(x^{-1})p(x) = p(x^{-1}x) = p(e) \in H\,.$$

A.2. Differential Geometrical Preliminaries (added as we move along, no logical order...)

LEMMA A.2.1. *Let M be a Riemannian manifold and $p_0 \in M$. Then there exists a ball $B_r(p_0)$ that is a normal neighborhood of each of its points, with the following property. Let $p, q \in B_r(p_0)$ and let $\gamma : [0,1] \to M$ the unique geodesic in $B_r(p_0)$ joining $p = \gamma(0)$ and $q = \gamma(1)$ and let $\mathrm{L}(\gamma)$ be its length. Then:*

(1) *For any (p', v) near $(p, \dot\gamma(0))$ with $v \in T_{p'}M$, $\|v\| = 1$, there exists a geodesic γ' in $B_r(p_0)$ of the same length as $\mathrm{L}(\gamma)$ starting at p' and with v as tangent vector in p'.*
(2) *The data (p', v) (and hence the geodesic γ') depends smoothly on $(p, \dot\gamma(0))$.*

A.2.1. Completeness.

THEOREM A.2.2 (Hopf–Rinow). *Let (M, g) be a Riemannian manifold. The following assertions are equivalent:*

(1) *M is geodesically compete, that is, all geodesics are defined over $\mathbb{R}$ or, equivalently, Exp_p is defined on T_pM for all $p \in M$;*
(2) *There exists $p \in M$ such that Exp_p is defined on T_pM;*
(3) *(M, d) is complete as a metric space.*
(4) *The closed bounded subsets of M are compact.*

Moreover, each of these assertions implies the existence of a minimizing geodesic between any two given points.

A.2.2. Connections.

DEFINITION A.2.3. A C^∞ or *affine connection* ∇ on a differentiable manifold (M, g) is a map $\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$, $(X, Y) \to \nabla_X Y$ with the properties that for every $f, f' \in C^\infty(M)$ and every $X, X', Y, Y' \in \mathrm{Vect}(M)$,

(1) $\nabla_{fX+f'X'} Y = f(\nabla_X Y) + f'(\nabla_{X'} Y)$;
(2) $\nabla_X(fY + f'Y') = f\nabla_X Y + f'\nabla_X Y' + (Xf)Y + (Xf')Y'$.

REMARK A.2.4. (1) A C^∞ connection ∇ is $\mathbb{R}$-linear in both variables, but it is $C^\infty(M)$-linear only in the first variable and not in the second one.

(2) Another difference between the role that the two variables X, Y play, is reflected in the fact that the value at the point $p \in M$ of the vector field $\nabla_X Y$ depends only on the value X_p of the vector field X at p, but not on the vector field X. (The same is not true of the dependence on Y.)

A connection allows to differentiate vector fields defined along curves. If $\gamma : \mathbb{R} \to M$ is a smooth curve, we call $\nabla_{\dot\gamma} X$ the *covariant derivative* of X along γ.

Definition A.2.5. We say that a vector field X along a curve γ is *parallel* if $\nabla_{\dot\gamma} X = 0$.

While "constant" vector fields, that is vector fields $Y \in \mathrm{Vect}(M)$ such that at every point $p \in M$ $(\nabla_X Y)_p = 0$ for every $X \in \mathrm{Vect}(M)$ rarely exists, it follows from the existence and uniqueness of the solutions of differential equations always exist:

Proposition A.2.6. *Let M be a differentiable manifold. Given a curve γ and a vector $v \in T_{\gamma(0)}M$, there exists a unique vector field X^v parallel along γ such that $X_{\gamma(0)} = v$.*

The *parallel transport* along γ from $\gamma(0)$ to $\gamma(t)$ is defined as the linear isomorphism $T_{\gamma(0)}M \to T_{\gamma(t)}M$ given by $v \mapsto (X^v)_{\gamma(t)}$. This gives an identification of the tangent spaces at $\gamma(0)$ and at $\gamma(t)$. Geodesics in a differentiable manifolds are defined as differentiable curves $\gamma : I \to M$ such that $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all $t \in I \subset \mathbb{R}$.

One could add more conditions to the ones defininining an affine connection. An affine connection satisfying also condition (3) below is called a symmetric connection.

Definition A.2.7. An affine connection that in addition satisfies

(3) it is *symmetric*, namely $[X, Y] = \nabla_X Y - \nabla_Y X$, and
(4) $Xg(Y, Y') = g(\nabla_X, Y, Y') + g(Y, \nabla_X Y')$

is called *Riemannian connection.*

Theorem A.2.8 (Fundamental Theorem in Riemannian Geometry). *Given a Riemannian manifold (M, g), there exists a unique Riemannian connection, called the Levi-Civita connection.*

The following lemma is not at all surprising:

Lemma A.2.9. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $\gamma : \mathbb{R} \to M$ be a smooth curve, Y a parallel vector field along γ and $f \in \mathrm{Iso}(M)$. Then $f_* Y$ is a parallel vector field along $f \circ \gamma$.*

Proof. Let us consider the map $\mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by $(X, Y) \mapsto f_*^{-1}(\nabla_{f_* X} f_* Y) =: D_X Y$. If we show that $D_X Y$ satisfies (1) through (4) of Definition A.2.3, then by Theorem A.2.8, $\nabla_X Y = f_*^{-1}(\nabla_{f_* X} f_* Y)$, so that $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$. If $X = \dot\gamma$, then $f_*(\nabla_{\dot\gamma} Y) = \nabla_{f_*\dot\gamma} f_* Y = \nabla_{(f\circ\gamma)^{\cdot}} f_* Y$, so that $\nabla_{(f\circ\gamma)^{\cdot}} f_* Y = 0$ if $\nabla_{\dot\gamma} Y = 0$.

Properties (1) and (2) are obvious. To see (3), recall that $[f_*X, f_*Y] = f_*[X,Y]$, so that

$$\nabla_{f_*X} f_*Y - \nabla_{f_*Y} f_*X = [f_*X, f_*Y] = f_*[X,Y] = f_*(\nabla_X Y - \nabla_Y X)\,.$$

It follows that

$$[X,Y] = \nabla_X Y - \nabla_Y X = D_X Y - D_Y X\,,$$

so that (3) is verified. Then (4) follows from the following chain of equalities:

$$\begin{aligned}
&g(D_X Y, Y') + g(Y, D_X Y')\\
=&g(f_*^{-1}(\nabla_{f_*X} f_*Y), Y') + g(Y, f_*^{-1}(\nabla_{f_*X} f_*Y')\\
=&g(\nabla_{f_*X} f_*Y, f_*Y') + g(f_*Y, \nabla_{f_*X} f_*Y')\\
=&(f_*X)g(f_*Y, f_*Y')\\
=&f_*X)f_*(g(Y,Y'))\\
=&Xg(Y,Y')\,.
\end{aligned}$$

$\square$

In fact, this is only a particular case of the fact that if $f : M \to M$ is a diffeomorphism and $\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is an affine connection, then $D : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by $D_X Y := f_*^{-1}(\nabla_{f_*X} f_*Y)$ is also an affine connection. In particular, if M is a Lie group and $f := L_g$ is the left translation via $g \in G$, then a connection that satisfies

$$\nabla_X Y := (L_g)_*^{-1}(\nabla_{(L_g)_*X}(L_g)_*Y) \tag{A.2.1}$$

is called *left invariant*.

Here is a result about the differential of the exponential map associated to an affine connection. Recall that an affine connection is *analytic* if the map $p \mapsto (\nabla_X Y)_p$ is analytic for any two analytic vector fields $X, Y \in \mathrm{Vect}(M)$.

If $X \in T_pM$, for $p \in M$, we denote by X^* the vector field defined on a normal neighborhood around $p \in M$, obtained by parallel translation of X along a geodesic joining two points.

Recall that there exist neighborhoods $N_q(M)$ of $q \in M$ and V_qM of $0 \in T_qM$ such that $\mathrm{Exp}_q : V_q \to N_qM$ is a diffeomorphism. The differential at $X \in V_qM$ will hence be: $d_X(\mathrm{Exp}_\nabla)_q : T_X(V_qM) \to T_{(\mathrm{Exp}_\nabla)_q(X)}N_q(M)$ or else, by identifying $T_X(V_qM)$ with T_qM,

$$d_X(\mathrm{Exp}_\nabla)_q : T_qM \to T_{(\mathrm{Exp}_\nabla)_q(X)}(M)\,.$$

An analogous relation holds for tX, provided that t is small enough that $tX \in V_qM$.

If $X \in T_qM$, for $q \in M$, we denote by X^* the vector field defined on a normal neighborhood around $q \in M$, obtained by parallel translation of X along a geodesic joining two points.

Theorem A.2.10 ([**Hel01**, Theorem I.6.5]). *Let M be an analytic manifold with an analytic connection. Let $q \in M$ and $X \in T_qM$. Then there exists $\epsilon > 0$ such that for $Y \in T_qM$,*

$$(d_{tX}(\mathrm{Exp}_\nabla))(Y) = \left(\sum_{n=0}^{\infty} \frac{\theta(-tX^*)^n}{(n+1)!}(Y^*) \right)_{(\mathrm{Exp}_\nabla)(tX)}$$

for $|t| < \epsilon$, where $\theta(X) := [X, Y]$.

A.2.3. Curvature. We know that if X, Y are vector fields, $[X, Y]$ measures the extent to which X and Y do not commute. We can also define a quantity that measures the extent to which $\nabla_X Y$ and $\nabla_Y X$ do not commute, by adding also a term that depends on $[X, Y]$ and "makes things better".

Definition A.2.11. Let M be a manifold with an affine connection. The *curvature* of M is a multilinear mapping (when $\mathrm{Vect}(M)$ is considered as a $C^\infty(M)$-module) $R : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by

$$R(X, Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}(Z) \, .$$

To every $X, Y \in \mathrm{Vect}(M)$, it associates the *curvature operator*

$$R(X, Y) : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$$

It follows from the presence of the term $\nabla_{[X,Y]}$ that at each point $p \in M$ the vector $(R(X, Y)Z)_p$ depends only on X_p, Y_p, Z_p and not on their values in a neighborhood of p. Thus R defines a linear transformation $R(X_p, Y_p) : T_pM \to T_pM$ and in fact $R : T_pM \times T_pM \to \mathrm{Lin}(T_pM)$ is a map that to two vectors at the point p, associates a linear operator from T_pM into itself.

If M is a Riemannian manifold, then the Riemannian metric allows us to see the curvature as a (4,0)-tensor, by setting $R(X, Y, Z, T) = g(R(X, Y)Z, T)$.

The Riemann curvature tensor has the following symmetries:

(R_1) $R(X, Y, Z, T) = -R(Y, X, Z, T) = R(Z, T, X, Y)$;
(R_2) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ (First Bianchi Identity)

It is not difficult to see that the curvature tensor and the curvature operator completely determine each other.

Given a Riemannian manifold, there are other notions of curvature. The *sectional curvature* $K(P)$ of a 2-plane P in T_pM is defined as follows. If $\{u, v\}$ is an orthonormal basis of P (orthonormal with respect to the Riemannian metric g) then

$$K(P) := -R(u, v, u, v) \, . \tag{A.2.2}$$

The sectional curvature coincides with the usual notion of Gaussian curvature on a surface. Namely, if P is a tangent 2-plane in T_pM and Σ is a portion of surface in M tangent to P at p, then the sectional curvature of P is exactly the Gaussian curvature of Σ at p.

Moreover the symmetry properties of the Riemann curvature tensor imply that it can be completely determined by knowing the sectional curvature on all sections of T_pM.

A.2.4. Totally Geodesic Submanifolds.

DEFINITION A.2.12. Let M be a Riemannian manifold and $N \subset M$ a connected submanifold. Let $p \in N$ The submanifold N is *geodesic at* p if given any tangent vector $v \in T_pN$, the M-geodesic $\gamma_v : -(\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$ is contained in N.

The submanifold N is *totally geodesic* if it is geodesic at every point $p \in N$.

It is not difficult to show that then the M-geodesic $\gamma \subset N$ is also an N-geodesic and that any N-geodesic is also an M-geodesic. As a consequence, if M is complete, then N is complete.

Totally geodesic submanifolds in Riemannian manifolds are not frequent. If $M = \mathbb{R}^n$, then linear subspaces and their translates are totally geodesic. If $M = S^n \subset \mathbb{R}^{n+1}$, then the intersection of S^n with linear subspaces are totally geodesic. It was proven by Cartan, that if a Riemannian manifold M has the property that for every $p \in M$ and for every two-dimensional plane $P \subset T_pM$, there exists a totally geodesic submanifold tangent to P, then M has constant curvature.

THEOREM A.2.13. *Let M be a Riemannian manifold and N a connected complete submanifold. Then N is totally geodesic if and only if the M-parallel transport along curves in N sends tangent vectors to N to tangent vectors to N.*

One direction of the above theorem is obvious if we replace "curve" with "geodesic".

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